Period Length and the Set of Dynamic Equilibria with Commodity Money*

Ezra Oberfield — University of Chicago — ezrao@uchicago.edu

Nicholas Trachter — EIEF — trachter@gmail.com

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Abstract

A property of the Kiyotaki and Wright (1989) model of commodity money as a medium of exchange is the multiplicity of dynamic equilibria as discussed in Kehoe et al. (1993). We adapt the model to allow the meeting rate to depend on the length of the time period and focus on symmetric dynamic equilibria in a symmetric environment. We characterize the set of points in the state-payoff space that are consistent with equilibrium. With a time period of any fixed length, there is a large set of equilibria that includes cycles, sunspots, and other non-Markovian strategies, while in the continuous time limit there is a unique, rather simple, dynamic equilibrium. Despite the multiplicity, for short period lengths all equilibrium paths are well approximated by the unique equilibrium of the continuous time limit.

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1 Introduction

In their seminal contribution, Kiyotaki and Wright (1989) introduced a search model of commodity money in which goods with no intrinsic value can function as a store of value, sparking a robust literature. Characterizing the dynamics of this model can be difficult because, as shown in Kehoe et al. (1993), the set of dynamic equilibria is large and includes cycles, sunspots, and other non-Markovian equilibria. The set of symmetric equilibria in a symmetric economy retains this multiplicity, but in the continuous time limit there is a unique dynamic equilibrium.

The fact that there are multiple equilibria for *any* strictly positive period length but a unique equilibrium in continuous time gives the impression that there is a qualitative difference between discrete and continuous time. Many of the recent papers in the literature start in continuous time, in part because the dynamics are significantly easier to characterize. If the relevant time period is indeed discrete, one might think that a continuous time approximation may be ignoring potentially interesting and relevant dynamics. We show that despite the qualitative difference, the unique equilibrium in continuous time is "close" to all of the equilibria in the discrete time model when the period length is short.

We describe the role of the length of the time period in generating multiplicity. We focus on symmetric equilibria in an economy with symmetric parameters and initial conditions, where both the discount factor and the meeting rate are adapted to depend on the length of the time period. We characterize the set of points in the state-payoff space that are consistent with a symmetric equilibrium. The size of this set varies directly with the probability of meeting a trading partner within a single period, decreasing monotonically as the period length shrinks.

We strengthen this result by showing that the set of equilibrium paths of the economy converges uniformly to the equilibrium path of the continuous time limit of the model, as do a local average of the actions played. This suggests that if the relevant period length is short, all dynamic equilibria are well approximated by the unique equilibrium of the continuous time limit.

In a search model of commodity money, multiplicity of dynamic equilibria arises from several sources. A natural source is asymmetric strategies potentially driven by differences in storage costs across goods.[1] A second source is an endogenous price level. Zhou (2003) shows that in a search model of commodity money, the price level is indeterminate when the utility value of the commodity money is small enough. The various equilibria that arise from these sources can be helpful in thinking about currency substitution patterns. In contrast to those arising from asymmetries, the equilibria arising from the length of the time period disappear when the economy is modeled in continuous time, and the relevance of the multiplicity shrinks with the period length.

In a dynamic rational expectations equilibrium, actions generate and must be consistent with the evolution of the economy. As time is divided into more subperiods, there are more restrictions on behavior in a given equilibrium as the set of choices that must be consistent with equilibrium expands. In the Kiyotaki and Wright (1989) model, this reduces the size of the set of potential equilibrium paths and payoffs.

In some contexts, dividing time into more subperiods can increase the set of feasible payoffs. The well known Folk theorem implies that decreasing period length can expand the set of possible equilibrium payoffs because the increased number of restrictions can reinforce cooperation. Faingold (2008) shows that in a game with imperfect monitoring, the set of possible payoffs available to a long-lived player increases as periods are divided into shorter subperiods, allowing the player to overcome commitment problems.

[1] Kehoe et al. (1993) discuss different steady states that can arise in the model. Renero (1998) discusses the stability of these steady states.

The role of the length of the time period has been studied in relation to cycles and sunspots in the real business cycle literature. In these models, when the period length is short enough cycles disappear. While superficially similar, the two models have disparate sources of multiplicity, and consequently the relationship between period length and multiplicity differ. Multiplicity in the RBC model stems from increasing returns or monopolistic competition (see Benhabib and Farmer (1999)). As the period length shrinks, there is a critical period length below which multiplicity disappears (see Boldrin and Montrucchio (1986) and Hintermaier (2005)). In contrast, in the Kehoe et al. (1993) model, there is a large multiplicity of equilibria for any positive period length. The multiplicity arises from the use of mixed strategies when agents are indifferent between two actions.

Cass and Shell (1983) and Azariadis and Guesnerie (1986) show that multiplicity and sunspots are present in OLG models, while Lomeli and Temzelides (2002) and Lagos and Wright (2003) argue that there is a strong relationship between these and search models. In a typical OLG model, as the period length decreases two things change: the length of the life of the agents and also the total number of transactions in a given unit of time. In contrast, in a search theoretic model one can change the length of the time period leaving the average number of transactions unchanged. In a sense the changes in the period length have different implications for the interpretation of the two types of models.

Section 2 sets up the economic environment while Section 3 describes symmetric equilibria. Section 4 gives examples of the many types of equilibria that can arise and shows that these exist for any finite period length. We characterize the set of perfect foresight equilibria in Section 5 and show how this varies with the period length. Section 6 extends these results to include sunspot equilibria.

2 Model

Time is discrete with h being the length of time elapsed between periods. There are three types of individuals, indexed by the type of good they consume. An individual of type i gets instantaneous utility u from consuming good i, and produces good $i+1 \pmod 3$ at zero cost.

Each period, an individual has a random encounter with another individual with probability $\alpha(h)$ and meeting each type of individual is equally likely. During this encounter, the two individuals can exchange goods and consume. Immediately after consumption the individual produces a new good at zero cost. The agent always stores a good at cost c per unit of time.

Even though there is never a double coincidence of wants, an individual may accept a good that she does not want to consume in order to exchange it later for the good she desires. In this way, the intermediate good acts as a store of value.

Individuals discount future flows with the discount factor $\beta(h) < 1$. We assume that $\beta(h)$ is strictly decreasing and that $\lim_{h\downarrow 0} \beta(h) = 1$. In addition, we assume that $\alpha(h)$ is strictly increasing, and that $\lim_{h\downarrow 0} \alpha(h) = 0$.

2.1 Strategies and Equilibrium

Let I be the set of types of individuals and $\mathbf{T} = \{nh\}_{n=0}^{\infty}$ denote the set of times at which transactions can take place. Let $p_t^i \in [0,1]$ be the fraction of individuals of type i that are storing good $i+1$ at time t, summarized by the vector $P_t \equiv \left\{p_t^i\right\}_{i\in I}$.

There may be equilibria in which individuals coordinate their actions using the realizations of a sunspot variable. Let $\{z_t\}_{t\in\mathbf{T}}$ be an extrinsic sequence of random variables that have no direct effect on the economic environment, and let $z^t = (z_0, ...z_t)$ be a history of realizations. The distribution of z_t can depend on both the history of realizations and t, but it turns out that the specific random process that drives z_t plays no role in our characterization of the set of equilibria.[2] Let $\mathbf{Z}^t$ be the support of z^t.

A strategy at time t is a function $\tau_t : I \times I \times I \times \mathbf{Z}^t \to [0, 1]$ that gives the probability that an individual is willing to exchange goods. $\tau_t(i, j, j', z^t)$ is the strategy for an individual of type i storing good j that meets another individual storing good j' at time t after the sunspot history z^t. We restrict strategies so that $\tau_t(i, j, j', z^t) = 1 - \tau_t(i, j', j, z^t)$, so that preferences over good j and j' are consistent at a given point in time.

Following the accounting convention of Kehoe et al. (1993), let $V_t^{i,j}$ denote the present discounted value for an individual of type i storing good j at the *end* of the period at time t.

$$V_t^{i,j}\left(z^t\right) = -ch + \max_{\{\tau_{t+nh}(\cdot)\}_{n=1}^{\infty}} \mathbb{E}\left\{ \sum_{n=1}^{\infty} \beta(h)^n \left(u\mathbb{I}_{t+nh}^u - ch\right) \middle| z^t \right\} \tag{1}$$

where $\mathbb{I}_t^u$ is an indicator that tells whether the individual consumes her good at time t and the expectation operator accounts for the the uncertainty of meeting trading partners and the possible realizations of the stochastic variable z_t. Note that an individual of type i always consumes good i as soon as she receives it.

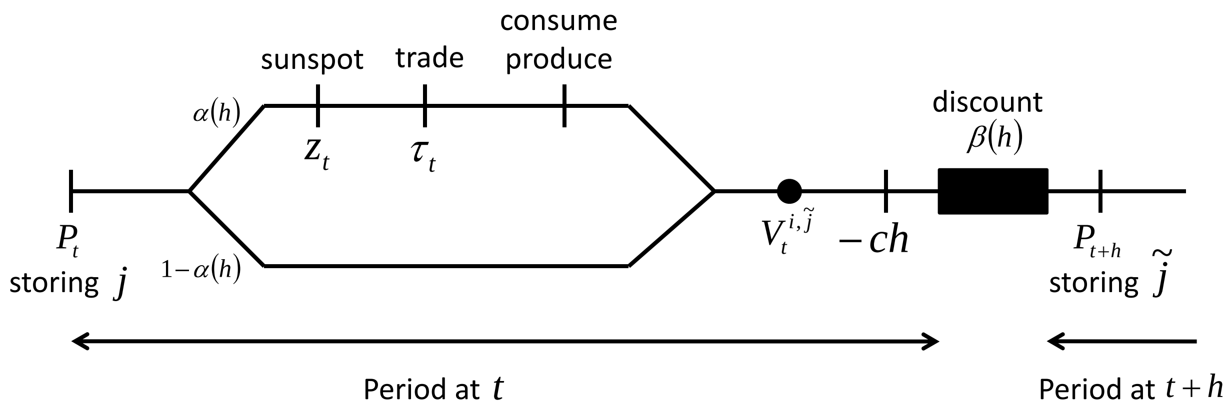
Figure 1 describes the timing of the environment and the accounting of the model.

Definition 1 *An Equilibrium is a sequence of strategies $\{\tau_t\}$ that satisfy (i) maximization: τ_t maximizes expected utility (equation (1)) given strategies of others and the distribution of inventories P_t, and (ii) rational expectations: given τ_t, P_t is the resulting distribution of inventories.*

When an individual meets a trading partner with the good she wants to consume, she will always

[2] Formally, let $z_t \sim \mu_t(\cdot; z^{t-h}, h)$.

Figure 1: Timeline



At the beginning of period t, P_t describes the distribution of inventories and individual i is holding good j. The individual meets a trading partner with probability $\alpha(h)$ and the sunspot variable z_t is realized. If she meets a trading partner, each chooses a trading strategy τ_t and trade may take place. If there is a trade, the individuals may consume and produce new goods. $V_t^{i,\tilde{j}}$ denotes the present discounted value at this point, where $\tilde{j}$ is the good that individual i is storing at the end of period t. The individual pays the storage cost and the period ends. Discounting occurs in between periods.

want to make the trade. When she meets a trading partner that is storing the same good as her, there are no gains from trade. Without loss of generality we set the trading strategy to be zero. The only strategy that is not pinned down immediately is whether to trade one type of good in order to store the other type of good. To simplify notation, let s_t^i be the probability that an individual of type i wants to exchange good $i+1$ for good $i+2$. Formally, $s_t^i\left(z^t\right) = \tau_t\left(i, i+1, i+2, z^t\right)$.

The probability of trade and the expected payoff from a meeting depend on the types of individuals that meet and the goods each is storing. Table 1 shows the strategies of both individuals and the potential payoffs for all possible relevant meetings.

We can use the probabilities of trade in Table 1 to produce an equation describing the evolution

Table 1: Strategies and Payoffs from Encounters

i holding $i+1$

Trading Partner	Strategy of i	Strategy of j	Probability of Trade	Conditional Payoff	Expected Payoff
$j=i+1$ holding $i+2$	s_t^i	1	s_t^i	$V_t^{i,i+2}-V_t^{i,i+1}$	$s_t^i\left(V_t^{i,i+2}-V_t^{i,i+1}\right)$
$j=i+1$ holding i	1	1	1	u	u
$j=i+2$ holding i	1	s_t^{i+2}	s_t^{i+2}	u	$s_t^{i+2}u$
$j=i+2$ holding $i+1$	0	0	0	0	0

i holding $i+2$

Trading Partner	Strategy of i	Strategy of j	Probability of Trade	Conditional Payoff	Expected Payoff
$j=i+1$ holding $i+2$	0	0	0	0	0
$j=i+1$ holding i	1	$1-s_t^{i+1}$	$1-s_t^{i+1}$	$u+V_t^{i,i+1}-V_t^{i,i+2}$	$\left(1-s_t^{i+1}\right)\left(u+V_t^{i,i+1}-V_t^{i,i+2}\right)$
$j=i+2$ holding i	1	1	1	$u+V_t^{i,i+1}-V_t^{i,i+2}$	$u+V_t^{i,i+1}-V_t^{i,i+2}$
$j=i+2$ holding $i+1$	$1-s_t^i$	1	$1-s_t^i$	$V_t^{i,i+1}-V_t^{i,i+2}$	$(1-s_t^i)\left(V_t^{i,i+1}-V_t^{i,i+2}\right)$

This table describes the strategies and payoffs from the perspective of an individual of type i. The top panel describes these when the individual is storing good $i+1$ while the bottom panel describes these when storing $i+2$. Column 1 lists the trading partner. Columns 2 and 3 give the strategies of the individual and the trading partner respectively. Column 4 gives the probability of trade, the product of columns 2 and 3. Column 5 gives the payoff to the individual if the trade happens. Column 6 gives the expected payoff to the individual from the encounter, the product of columns 4 and 5.

of inventories p_t^i as a function of the strategies chosen s_t^i,

$$\begin{aligned} p_{t+h}^i\left(z^t\right) &= p_t^i\left(z^{t-h}\right)\left[1-\frac{\alpha(h)}{3}p_t^{i+1}\left(z^{t-h}\right)s_t^i\left(z^t\right)\right] \\ &+\left(1-p_t^i\left(z^{t-h}\right)\right)\frac{\alpha(h)}{3}\left[\left(1-p_t^{i+1}\left(z^{t-h}\right)\right)\left(1-s_t^{i+1}\left(z^t\right)\right)+p_t^{i+2}\left(z^{t-h}\right)\right] \\ &+\left(1-p_t^i\left(z^{t-h}\right)\right)\frac{\alpha(h)}{3}\left[\left(1-p_t^{i+2}\left(z^{t-h}\right)\right)\left(1-s_t^i\left(z^t\right)\right)\right] \end{aligned} \tag{2}$$

The first term is the probability that an individual of type i was storing good $i+1$ at t and is still storing good $i+1$ at $t+h$. The second and third terms represent the probability that she was storing good $i+2$ at t and is now storing $i+1$ at $t+h$.

We can also rewrite the sequential problem in equation (1) with a recursive representation. For an individual of type i, there are two relevant cases, one for each good she can store. The value of storing good $i+1$ is,

$$\begin{aligned} V_t^{i,i+1}\left(z^t\right) &= \mathbb{E}\{-ch+\beta(h)V_{t+h}^{i,i+1}+\beta(h)\tfrac{\alpha(h)}{3}[p_{t+h}^{i+1}s_{t+h}^{i}(V_{t+h}^{i,i+2}-V_{t+h}^{i,i+1}) \\ &\quad +(1-p_{t+h}^{i+1})u+p_{t+h}^{i+2}s_{t+h}^{i+2}u]|z^t\} \end{aligned} \tag{3}$$

while the value of holding $i+2$ is

$$\begin{aligned} V_t^{i,i+2}\left(z^t\right) &= \mathbb{E}\left\{-ch+\beta(h)V_{t+h}^{i,i+2}+\beta(h)\tfrac{\alpha(h)}{3}\left[\left(1-p_{t+h}^{i+1}\right)\left(1-s_{t+h}^{i+1}\right)\left(u+V_{t+h}^{i,i+1}-V_{t+h}^{i,i+2}\right)\right.\right. \\ &\quad \left.\left. +p_{t+h}^{i+2}\left(u+V_{t+h}^{i,i+1}-V_{t+h}^{i,i+2}\right)+\left(1-p_{t+h}^{i+2}\right)\left(1-s_{t+h}^{i}\right)\left(V_{t+h}^{i,i+1}-V_{t+h}^{i,i+2}\right)\right]|z^t\right\} \end{aligned} \tag{4}$$

If holding good $i+1$ is more valuable than holding $i+2$, the strategy $s^i=0$ is optimal (and $s^i=1$ for the opposite case). If holding either good is equally valuable then any strategy can be optimal. Let $\Delta_t^i=V_t^{i,i+1}-V_t^{i,i+2}$ denote the difference in value between storing $i+1$ and storing $i+2$. The optimal trading strategy s_t^i can now be expressed as a function of Δ_t^i,

$$s_t^i\left(z^t\right)\in\begin{cases}\{0\} & \text{if } \Delta_t^i\left(z^t\right)>0\\ [0{,}1] & \text{if } \Delta_t^i\left(z^t\right)=0\\ \{1\} & \text{if } \Delta_t^i\left(z^t\right)<0\end{cases} \tag{5}$$

We now can refine the definition of an equilibrium.

Definition 2 *For an initial condition, P_0, an equilibrium is a sequence of inventories p_t^i, trading strategies s_t^i, and value functions $V_t^{i,i+1}, V_t^{i,i+2}$ denoted by*

$$\left\{p_t^i\left(z^{t-h}\right), s_t^i\left(z^t\right), V_{t-h}^{i,i+1}\left(z^{t-h}\right), V_{t-h}^{i,i+2}\left(z^{t-h}\right)\right\}_{t\in\mathbf{T}, i\in I}$$

such that (i) equation (2), equation (3), equation (4) and equation (5) are satisfied and (ii) the transversality conditions $\lim_{t\to\infty}\beta(h)^{t/h}V_t^{i,j}=0$ *holds for* $j\in\{i+1,i+2\}, i\in I$.

3 Symmetric Equilibria

We focus on symmetric equilibria and therefore we restrict the inventories so that $p_t^i = p_t$ for all i and trading strategies so that $s_t^i = s_t$. In this case, the evolution of inventories in equation (2) reduces to

$$\begin{aligned} p_{t+h}\left(z^t\right) &= p_t\left(z^{t-h}\right) - \tfrac{\alpha(h)}{3}p_t^2\left(z^{t-h}\right)s_t\left(z^t\right) \\ &\quad + \tfrac{\alpha(h)}{3}\left(1-p_t\left(z^{t-h}\right)\right)\left[2\left(1-p_t\left(z^{t-h}\right)\right)\left(1-s_t\left(z^t\right)\right)+p_t\left(z^{t-h}\right)\right] \end{aligned} \tag{6}$$

and the evolution of Δ_t is

$$\Delta_t\left(z^t\right) = \beta(h)\mathbb{E}\left\{\begin{array}{c} \Delta_{t+h} + \frac{\alpha(h)}{3}u[s_{t+h}-p_{t+h}] \\ -\frac{\alpha(h)}{3}[p_{t+h}s_{t+h}+2(1-p_{t+h})(1-s_{t+h})+p_{t+h}]\Delta_{t+h} \end{array}\middle| z^t\right\} \tag{7}$$

The following lemma will assist in the characterization of equilibria. Of particular use, we show that if $\{\Delta\}$ corresponds to value functions that satisfy the sequence problem, then it must have a uniform bound.

Lemma 1 *A sequence* $\left\{p_t\left(z^{t-h}\right), s_t\left(z^t\right), \Delta_{t-h}\left(z^{t-h}\right)\right\}_{z^t\in\mathbf{Z}^t, t\in\mathbf{T}}$ *represents a symmetric equilibrium if and only if (i) equation (5), equation (6), and equation (7) are satisfied and, (ii) there exists* $B>0$ *such that for every* t, $\Pr\left\{\left|\Delta_{t-h}\left(z^{t-h}\right)\right|\leq B\right\}=1$.

Proof. See Appendix A. ■

Lemma 1 implies that we can look for equilibria in the space $\left\{p_t\left(z^{t-h}\right), s_t\left(z^t\right), \Delta_{t-h}\left(z^{t-h}\right)\right\}_{z^t\in\mathbf{Z}^t, t\in\mathbf{T}}$.

3.1 Symmetric Steady State Equilibrium

In this section we show existence and uniqueness of symmetric steady state equilibria. Kehoe et al. (1993) shows that with asymmetric storage costs there are a finite number of steady state equilibria. We show that with symmetric costs there is a unique symmetric steady state.

In any steady state equilibrium $\left\{p_t\left(z^{t-h}\right), s_t\left(z^t\right), \Delta_{t-h}\left(z^{t-h}\right)\right\} = \{p_{ss}, s_{ss}, \Delta_{ss}\}$ for all $t \in \mathbf{T}$ and $z^t \in \mathbf{Z}^t$. In this case equation (7) can be rearranged to get

$$\Delta_{ss} = \frac{u\left[s_{ss} - p_{ss}\right]}{\frac{1-\beta(h)}{\beta(h)}\left[\frac{\alpha(h)}{3}\right]^{-1} + p_{ss} + 2(1-p_{ss})(1-s_{ss}) + p_{ss}}$$

Since the denominator is positive, the value of Δ_{ss} and hence the optimal trading strategy s_{ss} depend on the sign of $s_{ss} - p_{ss}$. Consider first the possibility that $\Delta_{ss} < 0$: this would imply $s_{ss} = 1$ and hence $\Delta_{ss} \geq 0$, a contradiction. Consider next $\Delta_{ss} > 0$: this would imply $s_{ss} = 0$ and hence $\Delta_{ss} \leq 0$, also a contradiction. The only remaining possibility is $\Delta_{ss} = 0$ which holds only if $s_{ss} = p_{ss}$ which would be consistent with the optimal choice of the trading strategy given in equation (5). Using the evolution of inventories, equation (6), together with $p_t = s_t = p_{ss}$ for all $t \in \mathbf{T}$ provides $p_{ss} = s_{ss} = \frac{2}{3}$.

3.2 The Zero Equilibrium

We next consider a special dynamic equilibrium and label it the the *zero equilibrium*. As we will show below, an equilibrium of this type will be the unique surviving equilibrium as the period length, h goes to zero. The strategies of the zero equilibrium will also be helpful in characterizing the set of equilibria for any fixed h.

For any h, there exists a unique equilibrium for which $\Delta_t\left(z^t\right) = 0$ for all t. This equilibrium is

Markovian, and the strategy played is always $s_t = p_t$. This conditions implies that the probability of trading for the desired good is independent of the good the agent is holding.[3] It is easy to see that equation (5) and equation (7) are both satisfied. For any initial condition p_0, one can find the sequence of inventories by iterating equation (6). Such a sequence of $\{p_t\left(z^{t-h}\right), s_t\left(z^t\right), \Delta_{t-h}\left(z^{t-h}\right)\}$ satisfies the conditions of Lemma 1 and is therefore an equilibrium.

4 Indeterminacy

Kehoe et al. (1993) show that in a model with asymmetric parameters and asymmetric strategies there is a large multiplicity of dynamic equilibria. Here we give examples to demonstrate that many of these equilibria are also present in an environment with symmetric parameters and even with the restriction of symmetric strategies and initial conditions.

First we show that there is a continuum of deterministic dynamic equilibria in the sense discussed in section 6 of Kehoe et al. (1993).[4] Given the initial condition p_0, choose any s_0. This gives p_h. From p_h there exists strategies consistent with equilibrium such that $\Delta_t = 0$ for all $t \in \mathbb{T}$ (the strategies of the zero equilibrium). Then, since $\Delta_0 = 0$ the choice of s_0 is optimal independent of the value of Δ_{-h}. Because the choice of s_0 was arbitrary, each different s_0 corresponds to a different dynamic equilibrium.

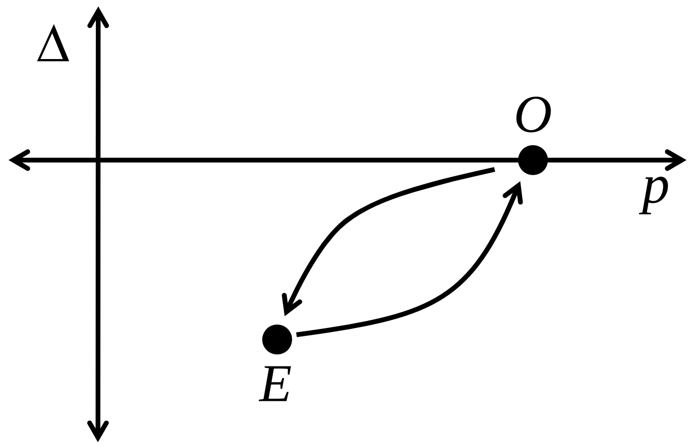
Second we can also construct cyclical equilibria. These are different from those discussed in Kehoe et al. (1993) as the ones we discuss are symmetric, but the idea is similar. We provide an example for the following parametrization: $\alpha(h) = 0.1$, $\beta(h) = 0.98$, and $u = 1$. The economy

[3] If individual 1 is holding good 2 her probability of trading for good 1 with a type 2 is $1 - p^2$ and with type 3 is $p^3 s^3$, so that the probability of trading for the desired good is $1 - p^2 + p^3 s^3$. If the individual 1 is holding good 3 her probability of trading for good 1 with a type 2 is $(1 - p^2)(1 - s^2)$ and with type 3 is p^3, so that the probability of trading for the desired good is $(1 - p^2)(1 - s^2) + p^3$. In a symmetric equilibrium these values are equated when $s = p$.

[4] The idea for this type of equilibrium originates with a construction by Aiyagari and Wallace (1992) with fiat money. Renero (1998) gives examples of equilibria of this type.

cycles between two triples $\{p_{nh}, s_{nh}, \Delta_{(n-1)h}\}$. When n is odd the economy lies at $\{0.6737, 1, 0\}$, and at $\{0.6659, 0.3174, -0.0114\}$ when n is even.

Figure 2: Example of a cyclical equilibrium



The equilibrium is characterized by $\{p_t, s_t, \Delta_{t-h}\}$. In this case: $E = \{0.6659, 0.3174, -0.0114\}$ and $O = \{0.6737, 1, 0\}$. Parametrization: $\alpha(h) = 0.1$, $\beta(h) = 0.98$ and $u = 1$.

Third we can also construct non-Markovian equilibria, combining the two previous examples. For the first $2N-1$ periods, individuals play the strategies associated with the cyclical equilibrium described above. From period $2N$ on, all individuals play the strategies associated with the zero equilibrium, so that $\Delta_t = 0$. In fact, we can construct an equilibrium in which every odd period there is a random variable that determines whether the individuals continue to play the cyclical strategies or the economy reverts to the zero equilibrium.

5 Perfect Foresight Equilibria

In this section we discuss perfect foresight equilibria in which the strategies played are independent of the realization of z_t. While individuals still face uncertainty in terms of meeting trading partners, p_t, s_t and Δ_t are no longer functions of z^t and follow deterministic paths. We can therefore drop the expectation operator in equation (7).

5.1 Continuous Time Limit of the Perfect Foresight Model

The dynamics of the continuous time limit of the model are simple and easy to describe. There is a unique equilibrium, in which agents choose $s \in (0,1)$ for all $t > 0$.

For the continuous time model to be well defined, we assume the following limits exist: Let $r = \lim_{h\downarrow 0} \frac{1}{h}\left(\frac{1}{\beta(h)} - 1\right)$ be the instantaneous discount rate and $\alpha_0 = \lim_{h\downarrow 0} \frac{\alpha(h)}{h}$ be the instantaneous meeting rate.

As $h \to 0$, equation (6) and equation (7) simplify to

$$\dot{p}_t = \frac{\alpha_0}{3}\left[-p_t^2 s_t + (1-p_t)\left(2(1-p_t)(1-s_t) + p_t\right)\right] \tag{8}$$

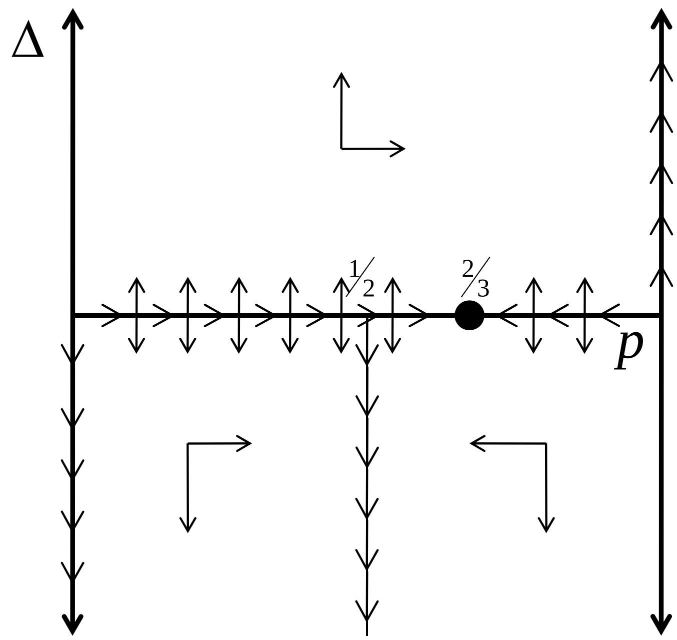
and

$$r\Delta_t = \dot{\Delta}_t + \frac{\alpha_0}{3} u\left[s_t - p_t\right] - \frac{\alpha_0}{3}\left[p_t s_t + 2(1-p_t)(1-s_t) + p_t\right]\Delta_t \tag{9}$$

It is straightforward to show the only symmetric equilibrium is the zero equilibrium, i.e., $\Delta_t = 0$ for all $t \geq 0$ and the optimal strategy must be $s_t = p_t$. One can extend the definition of symmetric equilibrium and Lemma 1 to the continuous case and show that for any equilibrium, $\{\Delta_t\}_{t\geq 0}$ must have a uniform bound. First, note that $\Delta_t > 0$ implies $\dot{\Delta}_t > r\Delta_t$ and similarly $\Delta_t < 0$ implies $\dot{\Delta}_t < r\Delta_t$. Together, these imply that if there is a t at which $\Delta_t \neq 0$ then $|\Delta|$ will grow exponentially and without bound, violating Lemma 1. Lastly, observe that if $\Delta_t = 0$, any strategy other than $s_t = p_t$ will push the economy away from $\Delta = 0$. These dynamics are summarized by the phase diagram in Figure 3.

Note also that the paths of p_t and Δ_t are continuous as the time derivatives of these objects are uniformly bounded. This is an important difference between discrete and continuous time as it restricts the acceptable strategies that are consistent with equilibrium.

Figure 3: Phase diagram for the model in its continuous time formulation



The unique equilibrium strategy sets $s_t = p_t$ such that $\Delta_t = 0$ for all t. The equilibrium converges to the unique steady state with $p_{ss} = \frac{2}{3}$.

5.2 Properties of the Set of Perfect Foresight Equilibria

In this section we will characterize the set of state-payoff combinations that are consistent with a symmetric equilibrium for a fixed period length h. In order to do this, it is helpful to discuss the timing of the model.

A strategy at a given point in time, s_t, affects both the fraction of individuals storing each type of good and the relationship between current and future present discounted values. Inspection of equation (6) and equation (7) reveals that s_{t+h} is relevant for the relationship between Δ_t and Δ_{t+h}

on the one hand, and p_{t+h} and p_{t+2h} on the other. In other words, s_{t+h} determines the relationship between (p_{t+h}, Δ_t) and (p_{t+2h}, Δ_{t+h}). Note that this is not an issue for the continuous time limit of the model.

We now characterize the set of points that are consistent with a symmetric equilibrium.

Proposition 1 *A sequence $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ that satisfies equation (5), equation (6), and equation (7) is an equilibrium* ***if and only if***

$$\Delta_t \in \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$$

where $\underline{\Delta}(p) = -\beta(h)\gamma(h)p$ and $\overline{\Delta}(p) = \beta(h)\gamma(h)(1-p)$ with $\gamma(h) = \frac{\alpha(h)}{3}u$.

Proof. See Appendix D. ■

Figure 4: Phase diagram for the model in its discrete time formulation for a fixed step size h

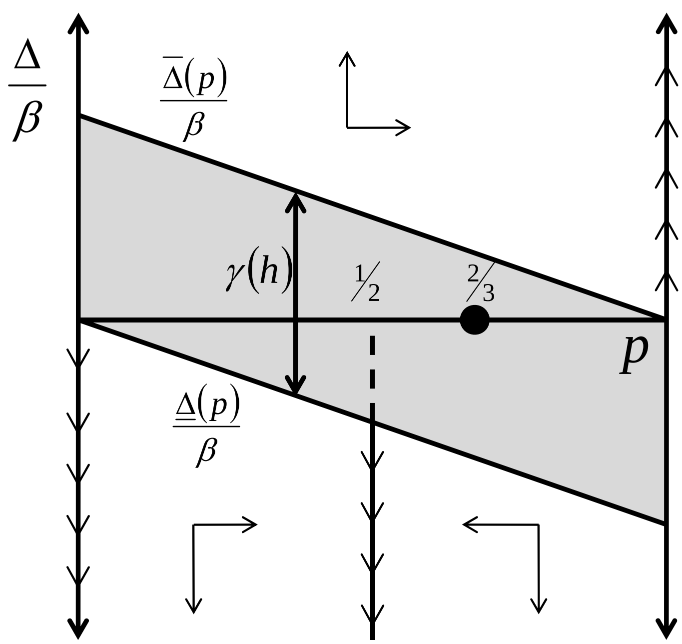


Figure 4, a partial phase diagram for a given length of period, h, gives a graphical representation of the main ideas in the proof of Proposition 1. Note that on the vertical axis we plot $\frac{\Delta}{\beta(h)}$. This corresponds to the value at the beginning of the next time period, so that both p_{t+h} and $\frac{\Delta_t}{\beta}$ refer to values at the beginning of period $t + h$.

The shaded area represents $\Gamma(h) \equiv \left\{(p, \frac{\Delta}{\beta(h)}) \text{ such that } \Delta \in \left[\underline{\Delta}(p), \overline{\Delta}(p)\right]\right\}$, the set of possible state-payoff combinations of $(p_{t+h}, \frac{\Delta_t}{\beta(h)})$ that are consistent with a symmetric equilibrium. One notable feature is that any point $(p, \frac{\Delta}{\beta})$ in the shaded area is consistent with an equilibrium in which (i) $p_{t+h} = p$ and $\Delta_t = \Delta$ and (ii) $\Delta_{t+h} = 0$. In other words, the economy can go from that point to $\Delta = 0$ in one period.

In fact, $\overline{\Delta}(p)$ is the upper bound on the set of Δ's such that the next period's Δ can be zero. This means that if Δ is above $\overline{\Delta}(p)$, we can guarantee that the strategy $s = 0$ is played. Using this we can show that the changes in Δ and p are both positive, which means that the next point in the sequence is also above $\overline{\Delta}(p)$ (this can be seen from the slope of $\overline{\Delta}(p)$). Because Δ is continually increasing, such a sequence will eventually violate the uniform bound implied by Lemma 1.

Similarly, below the $\underline{\Delta}(p)$, we can guarantee the strategy $s = 1$ is played. This implies that Δ the next period is more negative. For $p > 1/2$, p shrinks, implying that the next point in the sequence is also below $\underline{\Delta}(p)$. For $p < 1/2$, p rises, so from the phase diagram alone it is unclear whether the next point is below $\underline{\Delta}(p)$. However, in the proof we show algebraically that this is indeed the case. At one half, p is unchanged, but Δ is decreasing. Again, because Δ is continually becoming more negative, such a sequence would eventually violate the bound on Δ_t.

5.3 Period Length and the Set of Perfect Foresight Equilibria

The height of set of points consistent with symmetric equilibrium $\Gamma(h)$ is given by $\gamma(h) = \frac{\alpha(h)}{3}u$. As h decreases, the area of this set shrinks in proportion to $\alpha(h)$. In the limit, $\alpha(h)$, and hence $\gamma(h)$, approaches zero. In this case Figure 4 coincides exactly with the phase diagram of the continuous time model depicted in Figure 3. The only surviving equilibrium is the zero equilibrium.

The set of equilibrium points is increasing with $\alpha(h)$ because this set is determined by the range of values from which the next period's Δ can be equal to zero. For a given strategy, the expected change in value is increasing in the meeting rate, as the probability that the strategy is executed is higher. The larger the expected change in value, the larger the set of initial values that are consistent with a $\Delta_{t+h} = 0$. This can be seen easily by dividing equation (7) by $\alpha(h)$ and setting $\Delta_{t+h} = 0$.

We can also derive some properties of the sequence of inventories and trading strategies that are consistent with equilibrium. We show that the for any equilibrium, the sequence of inventories is "close" to that of the zero equilibrium. More formally, the set of sequences of inventories that are consistent with equilibrium converges uniformly to the sequence of inventories of the zero equilibrium.

Proposition 2 *For any $h > 0$ and p_0, let $\left\{p_t^0\right\}$ denote the sequence of inventories for the zero equilibrium. For any equilibrium, for all $t \in \mathbf{T}$*

$$\left|p_t - p_t^0\right| \leq \pi(h) \tag{10}$$

where $\lim_{h\to 0} \pi(h) = 0$.

Proof. See Appendix C.1. ■

After the previous proposition, it may not be surprising that the strategies played will also be "close" to those of the zero equilibrium. The next proposition shows that as h becomes small, the local average of the trading strategies converges to the strategies of the zero equilibrium.

Proposition 3 *For $\varepsilon > 0$, let N be the largest integer such that $\varepsilon \geq (2N+1)h$. Then in any equilibrium,*

$$\left| \left(\frac{1}{2N+1} \sum_{n=-N}^{N} s_{t+nh} \right) - p_t \right| \leq \sigma(h, \varepsilon) \tag{11}$$

holds for all $t \in \mathbf{T}$, with the property that $\lim_{\varepsilon \to 0} \left(\lim_{h \to 0} \sigma(h, \varepsilon) \right) = 0$

Proof. See Appendix C.2. ■

6 All Equilibria

The previous section discussed deterministic, perfect-foresight equilibria. Sunspots can occur if particular strategies that are chosen depend on random variables that have no intrinsic effect on the economy; individuals may use the realizations of the random variable to coordinate their strategies.

Remarkably the set of state-payoff combinations that are consistent with any equilibria coincides exactly with those of perfect foresight equilibria.

Proposition 4 *A sequence $\left\{ p_t \left(z^{t-h} \right), s_t \left(z^t \right), \Delta_{t-h} \left(z^{t-h} \right) \right\}_{z^t \in \mathbf{Z}^t, t \in \mathbf{T}}$ is consistent with equilibrium **if and only if***

$$\Pr \left\{ \Delta_t \left(z^t \right) \in \left[\underline{\Delta} \left(p_{t+h} \left(z^t \right) \right), \overline{\Delta} \left(p_{t+h} \left(z^t \right) \right) \right] \right\} = 1$$

where $\underline{\Delta}(p) = -\beta(h)\gamma(h)p$ and $\overline{\Delta}(p) = \beta(h)\gamma(h)(1-p)$ with $\gamma(h) = \frac{\alpha(h)}{3}u$

Proof. See Appendix D. ■

The idea behind the proof is similar to that of the perfect foresight case. We show that if Δ is above $\overline{\Delta}$ then we can guarantee that the strategy $s = 0$ is played with positive probability. With this, we can show if Δ is above $\overline{\Delta}$ with positive probability, then there must be a positive probability that the sequence of Δ's eventually violate the uniform bound given by Lemma 1. For any perfect foresight equilibrium, a special case, these positive probabilities are equal to 1.

We can also extend Proposition 2 and Proposition 3 to the set of all equilibria by adding expectations operators to the left hand sides of equation (10) and equation (11). [5]

7 Conclusion

In a model with commodity money as a medium of exchange, Kehoe et al. (1993) demonstrates the existence of a large set of dynamic equilibria. We have argued that the period length is a crucial determinant of the possible payoffs for any given initial condition. We analyze an economy with symmetric parameters and focus on symmetric strategies in order to highlight the role of the length of the time period. We characterize the set of state-payoff combinations are consistent with equilibrium and show that this set varies directly with the period length. If the probability of meeting another agent is proportional to the length of the time period, the set of state-payoff combinations is proportional to the period length. The continuous time limit of the model has a unique dynamic symmetric equilibrium with a simple characterization.

In some contexts multiplicity can give rise to vastly different long-run trajectories. We show

[5] One might think that would be possible to give a uniform bound $|p_t - p_t^0|$ for almost every z^t. However, one can find sunspot equilibria in which there is an arbitrarily small probability of an arbitrarily long sequence of any trading strategies, as long as the Δ at the end of the sequence is within the bounds at the end of the sequence. Because there were no restrictions on the sequence of trading strategies, there are no restrictions on p at the end of the sequence.

that multiplicity arising from the length of the time in Kehoe et al. (1993) model does not have this property when the period length is short. These arguments support the work that has followed Kiyotaki and Wright (1989) and uses continuous time.

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Appendix

A Proof of Lemma 1

We first show that for any equilibrium that satisfies the conditions of Definition 2, $\{\Delta_t\}_{z^t \in \mathbf{Z}^t, t \in \mathbf{T}}$ has a uniform bound with probability 1. It is straightforward to show that value functions $V_t^{i,j}(z^t)$ can be bounded above and below by bounds that are independent of z^t and t. For the upper bound, we can assume that the individual is able to consume at every chance meeting. For the lower bound we assume that the individual never consumes. The value function $V_t^{i,j}(z^t)$ can therefore be bounded by $-\frac{ch}{1-\beta(h)} \leq V^{i,j} \leq \frac{\alpha(h)u-ch}{1-\beta(h)}$. It follows that $\Delta_t(z^t)$ is bounded above and below by bounds that are independent of z^t and t. The other conditions of Definition 2 are trivially satisfied.

Second, we show that if a sequence $\{p_t, s_t, \Delta_{t-h}\}_{t \in \mathbf{T}}$ (i) satisfies equation (5), equation (6), and equation (7) and (ii) $\{\Delta_t(z^t)\}_{z^t \in \mathbf{Z}^t, t \in \mathbf{T}}$ is uniformly bounded, then we can construct a sequence of $\{p_t, s_t, V_{t-h}^{i,i+1}, V_{t-h}^{i,i+2}\}_{t \in \mathbf{T}}$ that is a symmetric equilibrium. We need to show that one can construct a sequence of value functions that satisfy the symmetric versions of equation (3) and equation (4) and transversality.

For a given sequence, define $M_t^1(z^t)$ and $M_t^2(z^t)$ to be

$$M_t^1(z^t) = -ch + \beta(h)\frac{\alpha(h)}{3}[p_t(z^{t-h})(-\Delta_t(z^t)) + (1 - p_t(z^{t-h}))u + p_t(z^{t-h})s_t(z^t)u]$$

and

$$\begin{aligned} M_t^2 \quad = \quad & -ch + \beta(h)\tfrac{\alpha(h)}{3}\left[\left(1 - p_t(z^{t-h})\right)\left(1 - s_t(z^t)\right)\left(u + \Delta_t(z^t)\right)\right. \\ & \left. + p_t(z^{t-h})\left(u + \Delta_t(z^t)\right) + \left(1 - p_t(z^{t-h})\right)\left(1 - s_t(z^t)\right)\left(\Delta_t(z^t)\right)\right] \end{aligned}$$

Iterating equation (3) and equation (4) and taking the limit as $N \to \infty$ gives

$$\begin{aligned} V_{-h}^{i,i+1} &= \mathbb{E}_{-h}\left[\textstyle\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^1\right] + \lim_{N\to\infty}\mathbb{E}_{-h}\left[\beta(h)^{N+1}V_{Nh}^{i,i+1}\right] \\ V_{-h}^{i,i+2} &= \mathbb{E}_{-h}\left[\textstyle\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^2\right] + \lim_{N\to\infty}\mathbb{E}_{-h}\left[\beta(h)^{N+1}V_{Nh}^{i,i+2}\right] \end{aligned}$$

where $\mathbb{E}_t(\cdot) = \mathbb{E}\left(\cdot \mid z^t\right)$. Since $\{\Delta_t(z^t)\}$ is uniformly bounded, the terms $\mathbb{E}_{-h}\left[\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^1\right]$ and $\mathbb{E}_{-h}\left[\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^2\right]$ are finite. If we set $V_0^{i,i+1} = -\mathbb{E}_{-h}\left[\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^1\right]$, then transversality must be satisfied. Since equation (3) and equation (4) are satisfied by construction, this is an equilibrium.

B Proof of Proposition 1

We develop the proof as a sequence of claims. Let $\{p_t, s_t, \Delta_{t-h}\}_{t \in \mathbf{T}}$ be a sequence that satisfies equation (5), equation (6), and equation (7).

Claim 1 *If $\Delta_t > \overline{\Delta}(p_{t+h})$ then $\Delta_{t+h} > \underline{\Delta}(p_{t+2h})$ and $\Delta_{t+h} \geq \Delta_t/\beta(h)$. Similarly, if $\Delta_t < \underline{\Delta}(p_{t+h})$ then $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$ and $\Delta_{t+h} \leq \Delta_t/\beta(h)$.*

Proof. Rearranging the perfect foresight version of equation (7) gives

$$\Delta_{t+h} = \frac{\Delta_t - \beta(h)\frac{\alpha(h)}{3}u(s_{t+h} - p_{t+h})}{\beta(h)\Omega_{t+h}}$$

where $\Omega_t = 1 - \frac{\alpha(h)}{3}\left[p_t s_t + 2(1-p_t)(1-s_t) + p_t\right] \in (0,1]$. $\Delta_t > \overline{\Delta}(p_{t+h})$ guarantees that $\Delta_{t+h} > 0$ and hence $s_{t+h} = 0$. Similarly, $\Delta_t < \underline{\Delta}(p_{t+h})$ guarantees that $\Delta_{t+h} < 0$ and hence $s_{t+h} = 1$.

Another rearrangement of equation (7) gives

$$\beta(h)\Delta_{t+h} - \Delta_t = -\beta(h)\frac{\alpha(h)}{3}u(s_{t+h} - p_{t+h}) + \beta(h)(1 - \Omega_{t+h})\Delta_{t+h}$$

If $\Delta_{t+h} > 0$, then $s_{t+h} = 0$ and hence $\beta(h)\Delta_{t+h} \geq \Delta_t$. Similarly, if $\Delta_{t+h} < 0$, then $s_{t+h} = 1$ and hence $\beta(h)\Delta_{t+h} \leq \Delta_t$.

We can also rearrange equation (6) to be

$$p_{t+2h} - p_{t+h} = \frac{\alpha(h)}{3}\left\{p_{t+h}(1 - 2p_{t+h})s_{t+h} + (2 - p_{t+h})(1 - p_{t+h})(1 - s_{t+h})\right\}$$

If $s_{t+h} = 0$ then $p_{t+2h} \geq p_{t+h}$. If $s_{t+h} = 1$ then the sign of $p_{t+2h} - p_{t+h}$ depends on whether $p_{t+h} \gtreqless 1/2$.

Consider first the case of $\Delta_t > \overline{\Delta}(p_{t+h})$. We have shown that $\Delta_{t+h} \geq \Delta_t$ and that $p_{t+2h} \geq p_{t+h}$. These, along with the fact that $\overline{\Delta}$ is decreasing in p imply that $\Delta_{t+h} > \overline{\Delta}(p_{t+2h})$.

Now consider $\Delta_t < \underline{\Delta}(p_{t+h})$. We have shown that $\Delta_{t+h} \leq \Delta_t$. If in addition $p_{t+h} \geq 1/2$, then $p_{t+2h} \leq p_{t+h}$. These along with the fact that $\overline{\Delta}$ is decreasing in p imply that $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$.

If, however, $p < 1/2$ then we cannot rely on this graphical argument because $p_{t+2h} > p_{t+h}$. Instead we check algebraically that $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$. We can write

$$\begin{aligned}
\frac{\Delta_{t+h} - \Delta_t}{p_{t+2h} - p_{t+h}} &= \frac{(1-\beta(h))\Delta_{t+h} - \beta(h)\left[\frac{\alpha(h)}{3}u(1-p_{t+h}) - \frac{\alpha(h)}{3}2p_{t+h}\Delta_{t+h}\right]}{\frac{\alpha(h)}{3}p_{t+h}(1-2p_{t+h})} \\
&< -\frac{\beta(h)u(1-p_{t+h})}{p_{t+h}(1-2p_{t+h})} \\
&< -\beta(h)u\frac{\alpha(h)}{3} \\
&= -\beta(h)\gamma(h)
\end{aligned}$$

where the last inequality follows as $p_{t+h} < \frac{1}{2}$ and $\frac{\alpha(h)}{3} < 1$.

Starting with $\Delta_t < -\beta(h)\gamma(h)p_{t+h}$, we have that

$$\begin{aligned}
\Delta_{t+h} &< -\beta(h)\gamma(h)p_{t+2h} + \beta(h)\gamma(h)p_{t+h} + \Delta_t \\
&< -\beta(h)\gamma(h)p_{t+2h} \\
&< \underline{\Delta}(p_{t+2h})
\end{aligned}$$

■

Claim 2 *$\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ are consistent with equilibrium if and only if $\Delta_t \in \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$ for all $t \in \mathbf{T}$*

Proof. If $\Delta_t \notin \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$ for some t, then the previous claim implies that $\Delta_{t+Nh} \notin \left[\underline{\Delta}(p_{t+(N+1)h}), \overline{\Delta}(p_{t+(N+1)h})\right]$ for all $N > 0$. Therefore $|\Delta_{t+Nh}| \geq \beta(h)^{-N}|\Delta_t|$. This would violate the uniform bound on $\{\Delta_t\}$, so the sequence cannot be an equilibrium.

If, however, $\Delta_t \in \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$ for all $t \in \mathbf{T}$ then the sequence $\{\Delta_t\}$ has a uniform bound. By Lemma 1, the sequence is consistent with equilibrium. ■

C Proofs of Proposition 2 and Proposition 3

We first prove a preliminary result that will help us prove Proposition 2 and Proposition 3. Let $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ be an sequence consistent with equilibrium.

Lemma 2 *For any $N > 0$, the following inequality holds:*

$$\sum_{n=1}^{N} \omega_{t,n,N} \left(s_{t+nh} - p_{t+nh}\right) \leq 2\frac{1-\beta(h)\left(1-\alpha(h)\right)}{1-\left[\beta(h)\left(1-\alpha(h)\right)\right]^N}$$

where

$$\omega_{t,n,N} = \frac{\prod_{j=1}^{n-1} \rho_{t+jh}}{\sum_{\tilde{n}=1}^{N}\left(\prod_{j=1}^{\tilde{n}-1} \rho_{t+jh}\right)}$$

and $\rho_t = \beta(h)\left(1 - \frac{\alpha(h)}{3}\left[p_t\left(1+s_t\right) + 2\left(1-p_t\right)\left(1-s_t\right)\right]\right)$.

Proof. equation (7) under perfect foresight can be written as

$$\Delta_t = \beta(h)\frac{\alpha(h)}{3}u\left(s_{t+h} - p_{t+h}\right) + \rho_{t+h}\Delta_{t+h}$$

We can iterate this equation to get

$$\Delta_t = \beta(h)\frac{\alpha(h)}{3}u\sum_{n=1}^{N}\left(\prod_{j=1}^{n-1}\rho_{t+jh}\right)\left(s_{t+nh} - p_{t+nh}\right) + \left(\prod_{n=1}^{N}\rho_{t+nh}\right)\Delta_{t+Nh}$$

where $\prod_{j=1}^{0}$ is defined to be 1.

Reordering terms, dividing by $\sum_{n=1}^{N}\left(\prod_{j=1}^{n-1}\rho_{t+jh}\right)$, and using the definition of $\omega_{t,n,N}$ provides

$$\beta(h)\frac{\alpha(h)}{3}u\sum_{n=1}^{N}\omega_{t,n,N}\left(s_{t+nh} - p_{t+nh}\right) = \frac{\Delta_t - \left(\prod_{n=1}^{N}\rho_{t+nh}\right)\Delta_{t+Nh}}{\sum_{n=1}^{N}\left(\prod_{j=1}^{n-1}\rho_{t+jh}\right)}$$

Since $\rho_t \in \left(\beta(h)\left(1 - \frac{2}{3}\alpha(h)\right), \beta(h)\right]$, we can bound the right hand side of this equation. The denominator is greater than $\sum_{n=1}^{N}\left[\beta(h)(1-\frac{2}{3}\alpha(h))\right]^{n-1}$, while the magnitude of the numerator is less than $2\beta(h)\gamma(h)$. These give the following bound:

$$\sum_{n=1}^{N} \omega_{t,n,N} \left(s_{t+nh} - p_{t+nh}\right) \leq 2 \frac{1 - \beta(h)\left(1 - \frac{2}{3}\alpha(h)\right)}{1 - \left[\beta(h)\left(1 - \frac{2}{3}\alpha(h)\right)\right]^N} \tag{12}$$

We can also use the bounds on ρ to bound each individual ω

$$\omega_{t,n,N} \in \left(\frac{\left[1 - \frac{2}{3}\alpha(h)\right]^N}{N}, \frac{1}{N\left[1 - \frac{2}{3}\alpha(h)\right]^N} \right) \tag{13}$$

■

C.1 Proof of Proposition 2

In any equilibrium, the sequence of inventories follows the equation

$$p_{t+h} = p_t + \frac{\alpha}{3}\left[-p_t^2 s_t + 2(1-p_t)^2(1-s_t) + p_t(1-p_t)\right]$$

Similarly, the sequence of inventories for the zero equilibrium must also follow the law of motion. Combining these equations give

$$p_{t+h} - p_{t+h}^0 = \Phi_t(s_t - p_t) - \lambda_t(p_t - p_t^0)$$

where Φ_t and λ_t are defined and bounded as follows

$$\Phi_t = 1 - \frac{\alpha}{3}\left\{-\left[3(p_t + p_t^0) - 4\right]p_t^0 - 3 + (p_t + p_t^0) + p_t^2 + 2(1-p_t)^2\right\} \in \left[1 - \frac{5}{3}\alpha, 1 - \frac{2}{3}\alpha\right] \tag{14}$$

and

$$\lambda_t = \frac{\alpha}{3}\left[p_t^2 + 2(1-p_t)^2\right] \in \left[\frac{2}{9}\alpha, \frac{2}{3}\alpha\right] \tag{15}$$

We can iterate this equation over N periods to get

$$p_{t+Nh} - p_{t+Nh}^0 = \sum_{n=0}^{N-1}\left(\prod_{j=n+1}^{N-1}\Phi_{t+jh}\right)\lambda_{t+nh}(s_{t+nh} - p_{t+nh}) + \left(\prod_{n=0}^{N-1}\Phi_{t+nh}\right)(p_t - p_t^0) \tag{16}$$

where again the product $\prod_{n=N}^{N-1}$ is defined to be one.

We now provide a bound on the divergence of inventories from those of the zero equilibrium among the first N periods. Since $p_0 = p_0^0$ and $|s_t - p_t| \leq 1$ we can use equation (16) and the upper bounds on Φ and λ given by equation (14) and equation (15) to get:

$$\left|p_{Nh} - p_{Nh}^0\right| \leq \frac{2}{3}\alpha \sum_{n=0}^{N-1}\left(1 - \frac{2}{3}\alpha\right)^n = 1 - \left[1 - \frac{2}{3}\alpha\right]^N$$

Define $\tilde{\pi}_0(h, N) \equiv 1 - \left[1 - \frac{2}{3}\alpha\right]^N$ to be this bound.

We next provide a bound on the subsequent divergence of inventories from those of the zero equilibrium. We can write equation (16) as

$$p_{t+Nh} - p^0_{t+Nh} = \left(1 - \prod_{n=0}^{N-1} \Phi_{t+nh}\right) \chi_{t,N} \sum_{n=0}^{N-1} \frac{\lambda_{t+nh}}{\lambda_t} \phi_{t,n,N}(s_{t+nh} - p_{t+nh}) + \left(\prod_{n=0}^{N-1} \Phi_{t+nh}\right)(p_t - p^0_t) \tag{17}$$

where χ and ϕ are defined by

$$\chi_{t,N} = \lambda_t \frac{\sum_{\tilde{n}=0}^{N-1}\left(\prod_{j=\tilde{n}+1}^{N-1} \Phi_{t+jh}\right)}{1 - \prod_{n=0}^{N-1} \Phi_{t+nh}}$$

$$\phi_{t,n,N} = \frac{\prod_{j=n+1}^{N-1} \Phi_{t+jh}}{\sum_{\tilde{n}=0}^{N-1}\left(\prod_{j=\tilde{n}+1}^{N-1} \Phi_{t+jh}\right)}$$

We will show that the term $\chi_{t,N} \sum_{n=0}^{N-1} \frac{\lambda_{t+nh}}{\lambda_t} \phi_{t,n,N}(s_{t+nh} - p_{t+nh})$ can be bounded by a function $\tilde{\pi}_1(h, N)$. This is useful because equation (17) would then imply that if $\left|p_t - p^0_t\right| \le \varepsilon$ for some $\varepsilon \ge \pi_1(N, h)$, then we also have $\left|p_{t+Nh} - p^0_{t+Nh}\right| \le \varepsilon$.

To do this, we first show that $|\chi_{t,N}| \le 1$. Since χ is increasing in each Φ_t, we can use the upper bounds on λ and Φ to get

$$|\chi_{t,N}| \le \left|\frac{2}{3}\alpha \frac{\sum_{n=0}^{N-1}\left(1 - \frac{2}{3}\alpha\right)^n}{1 - \left(1 - \frac{2}{3}\alpha\right)^N}\right| = 1$$

Next we can bound $\sum_{n=0}^{N-1} \frac{\lambda_{t+nh}}{\lambda_t} \phi_{t,n,N}(s_{t+nh} - p_{t+nh})$ by decomposing it into three parts using

$$\frac{\lambda_{t+nh}}{\lambda_t}\phi_{t,n,N} = \left(\frac{\lambda_{t+nh} - \lambda_t}{\lambda_t}\phi_{t+nh}\right) + (\phi_{t,n,N} - \omega_{t,n,N}) + (\omega_{t,n,N})$$

Using $|s_t - p_t| \le 1$ and $\phi_{t,n,N} > 0$ gives

$$\begin{aligned}\left|\sum_{n=0}^{N-1} \frac{\lambda_{t+nh}}{\lambda_t} \phi_{t,n,N}(s_{t+nh} - p_{t+nh})\right| \quad \le \quad & \sum_{n=0}^{N-1} \left|\frac{\lambda_{t+nh} - \lambda_t}{\lambda_t}\right| \phi_{t,n,N} + \sum_{n=0}^{N-1} |\phi_{t,n,N} - \omega_{t,n,N})| \\ & + \left|\sum_{n=0}^{N-1} \omega_{t,n,N}(s_{t+nh} - p_{t+nh})\right|\end{aligned}$$

We will bound each of these three terms separately. First, note that equation (6) implies

$$|p_{t+h} - p_t| = \frac{\alpha}{3}\left|(1 - 2p_t)\, p_t s_t + (2 - p_t)(1 - p_t)(1 - s_t)\right| \le \frac{2}{3}\alpha$$

and hence $|p_{t+nh} - p_t| \le n\left(\frac{2}{3}\alpha\right)$. We can also use the definition of λ to write

$$\left|\frac{\lambda_{t+nh} - \lambda_t}{\lambda_t}\right| = \left|\frac{\frac{\alpha}{3}(p_{t+nh} - p_t)(3(p_{t+nh} + p_t) - 4)}{\lambda_t}\right| \le \frac{\frac{\alpha}{3}\left(n\frac{2}{3}\alpha\right)4}{\frac{2}{9}\alpha} \le 4N\alpha$$

Since $\sum_{n=0}^{N-1} \phi_{t,n,N} = 1$, we have

$$\left| \sum_{n=0}^{N-1} \left| \frac{\lambda_{t+nh} - \lambda_t}{\lambda_t} \right| \phi_{t,n,N} \right| \leq 4N\alpha(h)$$

We can use the bound on Φ given by equation (14) to get upper and lower bounds for ϕ:

$$\phi_{t,n,N} \in \left(\frac{1}{N} \left(\frac{1 - \frac{5}{3}\alpha}{1 - \frac{2}{3}\alpha} \right)^N, \frac{1}{N} \left(\frac{1 - \frac{2}{3}\alpha}{1 - \frac{5}{3}\alpha} \right)^N \right)$$

This, in combination with the bounds on ω from equation (13) imply that

$$|\phi_{t,n,N} - \omega_{t,n,N}| \leq \frac{1}{N} \max_{\iota \in \{-1,1\}} \left| \left(1 - \frac{2}{3}\alpha\right)^{\iota N} - \left(\frac{1 - \frac{2}{3}\alpha}{1 - \frac{5}{3}\alpha} \right)^{\iota N} \right| = \left(1 - \frac{2}{3}\alpha\right)^N \left(\left(1 - \frac{5}{3}\alpha\right)^{-N} - 1 \right)$$

Lastly, the third term can be bounded using Lemma 2. In total, these give the result that

$$\chi_{t,N} \sum_{n=0}^{N-1} \frac{\lambda_{t+nh}}{\lambda_t} \phi_{t,n,N} (s_{t+nh} - p_{t+nh}) \leq \tilde{\pi}_1(h, N)$$

with

$$\tilde{\pi}_1(h, N) \equiv 4N\alpha(h) + \left(1 - \frac{2}{3}\alpha(h)\right)^N \left(\left(1 - \frac{5}{3}\alpha(h)\right)^{-N} - 1 \right) + 2 \frac{1 - \beta(h)\left(1 - \frac{2}{3}\alpha(h)\right)}{1 - \left[\beta(h)\left(1 - \frac{2}{3}\alpha(h)\right)\right]^N}$$

At this point we have shown that for any N, inventories in the first N periods are within $\tilde{\pi}_0(h, N)$ of those of the zero equilibrium. We have also shown that if inventories in the first N periods are within ε of those of zero equilibrium for any quantity $\varepsilon \geq \tilde{\pi}_1(h, N)$, then inventories in all subsequent periods are as well. We can combine these two statements to arrive at a uniform bound for the entire sequence. Define $\tilde{\pi}(h, N) = \max\{\tilde{\pi}_0(h, N), \tilde{\pi}_1(h, N)\}$. We therefore have that for any $N > 0$ and any $t \in \mathbf{T}$, inventories are within $\tilde{\pi}(N, h)$ of those of the zero equilibrium:

$$\left| p_t - p_t^0 \right| \leq \tilde{\pi}(h, N)$$

Let $\pi(h) = \min_N \tilde{\pi}(h, N)$. This will be a bound for $\left| p_t - p_t^0 \right|$.

Lastly, we can show that $\lim_{h \to 0} \pi(h) = 0$. Let $\nu(h) = h^{-1/2}$. From the definitions of $\tilde{\pi}_0$ and $\tilde{\pi}_1$ it is straightforward to show that $\lim_{h \to 0} \tilde{\pi}_0(h, \nu(h)) = \lim_{h \to 0} \tilde{\pi}_1(h, \nu(h)) = 0$. Since $\pi(h) \leq \tilde{\pi}(h, \nu(h))$, these imply that $\lim_{h \to 0} \pi(h) = 0$.

C.2 Proof of Proposition 3

In a similar way, we can show that, at least locally, the average trading strategy played coincides with that of the zero equilibrium.

For $\varepsilon > 0$, let N be the largest integer such that $\varepsilon \geq (2N + 1)h$. We can form a bound on the local average trading strategy:

$$\begin{aligned}\left|\left(\frac{1}{2N+1}\sum_{n=-N}^{N}s_{t+nh}\right)-p_t\right| \leq & \left|\sum_{n=-N}^{N}\left(\frac{1}{2N+1}-\omega_{t-N,n+N,2N+1}\right)(s_{t+nh}-p_{t+nh})\right| \\ & +\left|\sum_{n=-N}^{N}\omega_{t-N,n+N,2N+1}(s_{t+nh}-p_{t+nh})\right| \\ & +\left|\frac{1}{2N+1}\sum_{n=-N}^{N}(p_{t+nh}-p_t)\right|\end{aligned}$$

The first sum can be bounded using the bound on ω given by equation (13)

$$\begin{aligned}\left|\sum_{n=-N}^{N}\left(\frac{1}{2N+1}-\omega_{t-N,n+N,2N+1}\right)(s_{t+nh}-p_{t+nh})\right| \leq & \sum_{n=-N}^{N}\left|\frac{1}{2N+1}-\omega_{t-N,n+N,2N+1}\right| \\ \leq & \left(1-\frac{2}{3}\alpha\right)^{-(2N+1)}-1\end{aligned}$$

The second summation can be bounded using equation (12). The third term can be bounded using the fact that $|p_{t+h}-p_t| \leq \alpha(h)$, which can be seen from equation (6). This implies that

$$\left|\frac{1}{2N+1}\sum_{n=-N}^{N}(p_{t+nh}-p_t)\right| \leq N\alpha(h)$$

We can combine these to form a single bound for a fixed ε:

$$\sigma(\varepsilon,h)=\left(1-\frac{2}{3}\alpha\right)^{-\varepsilon/h}-1+2\frac{1-\beta(h)\left(1-\frac{2}{3}\alpha(h)\right)}{1-\left[\beta(h)\left(1-\frac{2}{3}\alpha(h)\right)\right]^{\varepsilon/h}}+\frac{\varepsilon}{2}\alpha(h)$$

For a fixed ε, each of these three bounds goes to a finite number as $h \to 0$:

$$\lim_{h\to 0}\sigma(\varepsilon,h)=e^{\frac{2}{3}\alpha_0\varepsilon}-1+\frac{\varepsilon}{2}\alpha_0$$

It follows that $\lim_{\varepsilon\to 0}(\lim_{h\to 0}\sigma(\varepsilon,h))=0$.

D Proof of Proposition 4

We develop the proof as a sequence of claims. For ease of exposition we drop the argument z from p_t, s_t, and Δ_t.

Let $\{p_t, s_t, \Delta_{t-h}\}_{z^t\in\mathbf{Z}^t, t\in\mathbf{T}}$ be a sequence that satisfies equation (5), equation (6), and equation (7). Also, Let $G_{t,n}$ be the event that $\Delta_{t-jh} \notin [\underline{\Delta}(p_{t-(j-1)h}), \overline{\Delta}(p_{t-(j-1)h})]$ for all $j \in (0,...,n)$. We can make the following claims about the sequence:

Claim 3 *If* $\Pr(G_{t,n})>0$ *then* $\Pr\left(G_{t+h,n+1} \text{ and } |\Delta_{t+h}| \geq \frac{|\Delta_t|}{\beta(h)}\right)>0$

Proof. The following definitions will assist in the exposition of the proof. As before, let $\Omega_t = \frac{\alpha(h)}{3}\left[p_t s_t + 2(1-p_t)(1-s_t) + p_t\right]$. Note that $\Omega_t \in [0,1]$. Also let $X_{t+h} = -\Delta_t + \beta(h)\Delta_{t+h} + \beta(h)\frac{\alpha(h)}{3}(s_{t+h} - p_{t+h})u - \beta(h)\Omega_{t+h}\Delta_{t+h}$. equation (7) can be rewritten as $0 = \mathbb{E}_t\left[X_{t+h}\right]$, where $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot \mid z^t)$. This implies both that $\Pr\left(X_{t+h} \geq 0 | z^t\right) > 0$ and also that $\Pr\left(X_{t+h} \leq 0 | z^t\right) > 0$ for all z^t. We therefore have that if $\Pr\left(G_{t,n}\right) > 0$ then either $\Pr\left(G_{t,n} \text{ and } \Delta_t > \overline{\Delta}(p_{t+h}) \text{ and } X_{t+h} \geq 0\right) > 0$ or $\Pr\left(G_{t,n} \text{ and } \Delta_t < \underline{\Delta}(p_{t+h}) \text{ and } X_{t+h} \leq 0\right) > 0$. We will show that in either case

$$\Pr\left(G_{t+h,n+1} \text{ and } |\Delta_{t+h}| \geq \frac{|\Delta_t|}{\beta(h)}\right) > 0$$

.

First, consider the event in which $\Delta_t > \overline{\Delta}(p_{t+h})$. If $\Delta_{t+h} \leq 0$, then it must be that $X_{t+h} < 0$, because $\Delta_t > \overline{\Delta}(p_{t+h}) \geq \beta(h)\frac{\alpha(h)}{3}(s_{t+h} - p_{t+h})u$. Consequently, if $X_{t+h} \geq 0$, then $\Delta_{t+h} > 0$ and therefore $s_{t+h} = 0$. The combination of $X_{t+h} \geq 0$ and $s_{t+h} = 0$ imply that $\Delta_{t+h} \geq \frac{\Delta_t}{\beta(h)}$ and $p_{t+2h} > p_{t+h}$. Since $\overline{\Delta}(p)$ is decreasing in p, these also imply that $\Delta_{t+h} > \overline{\Delta}(p_{t+2h})$. We therefore have that in the event that $\Delta_t > \overline{\Delta}(p_{t+h})$ and $X_{t+h} \geq 0$, then $\Delta_{t+h} \notin [\underline{\Delta}(p_{t+2h}), \overline{\Delta}(p_{t+2h})]$ and $|\Delta_{t+h}| \geq \frac{|\Delta_t|}{\beta(h)}$.

Now we turn to the event in which $\Delta_t < \underline{\Delta}(p_{t+h})$, $X_{t+h} \leq 0$, and $\Delta_{t+h} < 0$. We will show that in this case $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$. This is more difficult because the change in p is not a monotonic function of p. If $\Delta_t < \underline{\Delta}(p_{t+h})$ then in a similar manner as above we can show that $\Delta_{t+h} < 0$ and $s_{t+h} = 1$. This means that we can write

$$\Delta_t \geq \beta(h)\left[\Delta_{t+h} + \frac{\alpha(h)}{3}u(1 - p_{t+h}) - \Omega_{t+h}\Delta_{t+h}\right]$$

and

$$p_{t+h} = p_t + \frac{\alpha(h)}{3}p_t(1 - 2p_t)$$

We take two cases separately. For each we will show that if Δ_t is below the bound, than Δ_{t+h} is below the bound as well. (i) If $p_{t+h} \geq \frac{1}{2}$, then we can show this in a similar manner as above. Since $p_{t+2h} \leq p_{t+h}$ and $\Delta_{t+h} < \Delta_t < 0$, the fact that $\underline{\Delta}(p)$ is decreasing in p implies that $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$. (ii) If $p_{t+h} < 1/2$ then we can write

$$\begin{aligned}
\frac{\Delta_{t+h} - \Delta_t}{p_{t+2h} - p_{t+h}} &\leq \frac{(1-\beta(h))\Delta_{t+h} - \beta(h)\left[\frac{\alpha(h)}{3}u(1-p_{t+h}) - \Omega_{t+h}\Delta_{t+h}\right]}{\frac{\alpha(h)}{3}p_{t+h}(1-2p_{t+h})} \\
&< -\frac{\beta(h)u(1-p_{t+h})}{p_{t+h}(1-2p_{t+h})} \\
&< -\beta(h)u\frac{\alpha(h)}{3} \\
&= -\beta(h)\gamma(h)
\end{aligned}$$

where the last inequality follows because $p_{t+h} < \frac{1}{2}$ and $\frac{\alpha(h)}{3} < 1$. We start with $\Delta_t < -\beta(h)\gamma(h)p_{t+h}$.

We then have that

$$\begin{aligned}\Delta_{t+h} &< -\beta(h)\gamma(h)p_{t+2h}+\beta(h)\gamma(h)p_{t+h}+\Delta_t \\ &< -\beta(h)\gamma(h)p_{t+2h} \\ &< \underline{\Delta}(p_{t+2h})\end{aligned}$$

For both cases we also know that $X_{t+h}\leq 0$. This, in combination with $s_{t+h}=1$, implies that $\Delta_{t+h}\leq\frac{\Delta_t}{\beta(h)}$.

If $\Delta_{t+h}\geq 0$ then we know that $X_{t+h}>0$ because $\Delta_t<\underline{\Delta}(p_{t+h})\leq\beta(h)\frac{\alpha(h)}{3}(s_{t+h}-p_{t+h})u$. This implies that if $X_{t+h}\leq 0$, then $\Delta_{t+h}<0$. We have therefore shown that if $\Delta_t<\underline{\Delta}(p_{t+h})$ and $X_{t+h}\leq 0$, then $\Delta_{t+h}\notin[\underline{\Delta}(p_{t+2h}),\overline{\Delta}(p_{t+2h})]$ and $|\Delta_{t+h}|\geq\frac{|\Delta_t|}{\beta(h)}$.
■

Claim 3 shows that if $\Delta_t\notin\left[\underline{\Delta}(p_{t+h}),\overline{\Delta}(p_{t+h})\right]$, then the following Δ is outside the bounds with positive probability and the magnitude grows exponentially.

Claim 4 *If* $\Pr\left(\Delta_t\notin[\underline{\Delta}(p_{t+h}),\overline{\Delta}(p_{t+h})]\right)>0$ *then* $\left\{p_t\left(z^{t-h}\right),s_t\left(z^t\right),\Delta_{t-h}\left(z^{t-h}\right)\right\}_{z^t\in\mathbf{Z}^t,t\in\mathbf{T}}$ *is not consistent with equilibrium.*

Proof. Let B be the uniform bound implied by Lemma 1. Assume that there exists a t_0 such that $\Pr\left(\Delta_{t_0}\notin[\underline{\Delta}(p_{t_0+h}),\overline{\Delta}(p_{t_0+h})]\right)>0$. This implies that there exists $\epsilon>0$ such that $\Pr\left(\Delta_{t_0}\notin[\underline{\Delta}(p_{t_0+h}),\overline{\Delta}(p_{t_0+h})],|\Delta_{t_0}|>\epsilon\right)>0$. Iterating Claim 3 gives the result

$$\Pr\left(\Delta_{t_0+nh}\notin[\underline{\Delta}(p_{t_0+(n+1)h}),\overline{\Delta}(p_{t_0+(n+1)h})],|\Delta_{t_0+nh}|\geq\frac{|\Delta_{t_0}|}{\beta(h)^n}\geq\frac{\epsilon}{\beta(h)}\right)>0$$

Since there exists an $N>0$ such that $\frac{\epsilon}{\beta(h)^N}>B$, we have $\Pr\left(|\Delta_{t+Nh}|>B\right)>0$. ■