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PRELIMINARY DRAFT

Abstract

We study the stylized problem of a multi-product firm that can revise prices only after paying a fixed "menu" cost. The key assumption, introduced by Lach and Tsiddon (1996, 2007) and Midrigan (2007, 2009), is that once the menu cost is paid the firm can adjust the price of *all* its products. The firm's problem is to minimize the deviations of the profits incurred relative to the flexible price case, i.e. the case with no menu cost. We completely characterize the solution of a simple symmetric problem in terms of the structural parameters: the variability of the flexible prices, the curvature of the profit function, the size of the menu cost, and the number of products sold by the firm. We also provide analytical expressions for the frequency of adjustment, the hazard rate of price adjustments, and the distribution of price changes in terms of the structural parameters.

JEL Classification Numbers:

Key Words: menu cost, economies of scope in price changes, optimal control in multiple dimensions subject to fixed cost, quasi-variational inequalities

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1 Introduction

Several papers document how price setting behavior, as summarized by the size-distribution and by the timing of price changes, varies systematically with the number of products sold. The recent empirical work of Bhattarai and Schoenle (2010) documents that firms selling more goods display a higher frequency of price adjustment as well as smaller adjustments. Lach and Tsiddon (1996, 2007) show that price changes are synchronized within stores, but staggered across stores. Cavallo (2010) studies online large supermarket chains, and finds that price changes of similar goods are synchronized within a store.[1]

Despite the rich and growing evidence on this phenomenon, there is scant theoretical work on this problem. Midrigan (2007, 2009) begun to study this problem by explicitly writing down and solving numerically a model where a firm is selling 2 goods, subject to a common menu cost.[2] Compared to the classic case of one good, his model generates a distribution of price changes with "small" price adjustments. Indeed the main motivation on the seminal paper by Lach and Tsiddon (1996) was to argue that, due to the synchronization of adjustments, the presence of small price changes does not imply that menu cost are not important. But some important questions remain to be answered: what forces shape the optimal pricing decisions as the number of goods n sold by the firm changes? Going beyond the $n = 2$ case is important, as the number of goods sold by the retail stores, where much of the micro data are measured, is much larger.[3]

This paper provides a simple model to study how the price setting decision depends on n, the number of goods sold by a firm. The model allows us to answers some questions that are hard to tackle without a formal frame. Examples of these questions are: Is the higher

[1] An incomplete list of additional contributions documenting these type of behaviour includes Lach and Tsiddon (1992), Baudry et al. (2007), Dhyne and Konieczny (2007), Dutta et al. (1999), Midrigan (2007, 2009), and Neiman (2010).

[2] Importantly, he also put this model into a general equilibrium framework, and analyzed the effect of a monetary shock. Bhattarai and Schoenle (2010) also solve numerically the problem for a firm selling three goods.

[3] Bhattarai and Schoenle (2010) analyze the BLS data on US producer's prices. The median number of goods *sampled* by the BLS for each producer is between 3 to 5. Obviously this is a lower bound of the median number of goods sold by each firm.

frequency of price adjustment the mechanical consequence of a fixed menu ψ cost split over a larger number of goods, so that it is as if each good pays a smaller menu cost ψ/n as n increases? In other words, how does the frequency of price adjustment behaves as a function of n if the menu cost paid by the firm increased linearly with the number of goods: $n\ \psi$? Is the synchronization of price adjustment arising from the fact that a single fixed cost applies to a bundle of goods, or is there a systematic association between the number of products and the volatility σ of the shock that hit each product? Does the shape hazard rate of price adjustments varies systematically with the number of products? Does the shape of the distribution of price changes varies systematically with the number of products? One purpose of this note is to provide some guidance for the empirical examination of these concerns by analyzing the identification of the different factors, such as the n, the volatility of the determinant of prices, the size of the fixed cost, and the benefit of adjusting prices in a tractable set up. Another purpose of the note is to advance on the understanding of the impact effect of an aggregate monetary shock in this set-up. This question that has been tacked numerically by Midrigan (2007) for the case of $n = 2$, but as we explain below we provide tools that help to analyze in the case of $n > 2$.

We study a stylized version of the problem of a multi-product firm that can revise prices only after paying a fixed cost. The key assumption, introduced by Lach and Tsiddon (1996, 2007) and Midrigan (2007, 2009), is that once the fixed menu cost is paid the firm can adjust the price of *all* its products. The problem is set up as to minimize the deviations of the profits incurred relative to the flexible price case, i.e. the case with no menu cost. We assume that the static profit maximizing price for each of the n products, which coincide with the price that will be charged without menu cost, follows n independent random walks without drift and with volatility σ per unit of time. We refer to the vector of the difference between the frictionless prices and the actual prices charged as the vector of the price gaps. The period return function is assumed to be proportional to the sum of the squares of the price gaps. The proportionality constant B measures the second order per period losses associated with charging a price different from the optimum, i.e. it is a measure of the curvature of the profit function.[4] We assume that if a fixed cost ψ is paid the firm can simultaneously change all the prices. The firm minimizes the expected discounted cost, which include the stream of lost profit from charging prices different from the frictionless as well as the fixed cost at the time of adjustments. We completely characterize the solution of a simple symmetric problem in terms of the structural parameters: the variability of the flexible prices σ, the curvature of the profit function B, the size of the menu cost ψ, the discount rate r, and the number of products n. We also provide analytical expressions for the invariant distribution of the price gaps, the frequency of adjustment, the hazard rate of price adjustments, and the marginal distribution of price changes in terms of the fundamental parameters.

The solution of the firms' problem involves finding the set over which prices are adjusted, and the set where they are not, i.e. the inaction set. Due to the lack of drift, when prices are adjusted they are set equal to the frictionless prices, i.e. the price gaps are set to zero in all dimensions. We show that the optimal decision is to control the price gap as to remain in the interior of the n-dimensional ball centered at the origin. The economics of this is clear: the firm will adjust either if all the prices of its product have a medium size deviation, or if only one has a large deviation, since in the margin a larger deviation hurst profits more. The size of this ball, whose square radius we denote by $\bar{y}$, is chosen optimally. We solve for the value function and completely characterize the size of the inaction set $\bar{y}$ as a function of the parameters of the problem. As we let $r \downarrow 0$ the ratio $\bar{y}/\sigma^2$ can be written as an increasing function of two arguments: $\sigma^2 B/\psi$ and n. We also obtain a very accurate approximation for small cost ψ, where we show that $\bar{y}$ takes the form of a square root function, $\bar{y} \approx [2(n+2)\sigma^2 B/\psi]^{1/2}$. To compare the model with tabulation for the US economy as functions of n, we consider two extreme cases of how the the technology to adjust prices. In one case we assume that the fixed cost increases proportionally with the number of products, i.e. $\psi = \psi_1 \, n$ for some $\psi_1 > 0$, a case that we referred to as *constant returns*

[4] The first order losses are zero, since the maximum per period profits are obtained at the frictionless price.

to scale. In the other extreme the fixed cost remains constant as n changes, an assumption that we referred to as *constant fixed cost*, so $\psi = \psi_1$ for all n. Thus, when a prediction of the model depends on how the technology varies across n we present both cases.

To our knowledge this is the first fixed cost adjustment problem in n-dimensions whose solution is analytically characterized. We believe that this is because of the difficulty of finding a tractable boundary condition and a candidate solution that is smooth enough on the boundary of the inaction region. Baccarin (2009) gives a recent statement of the general problem, and an existence results of a viscosity solution. Instead we look for a strong solution, i.e. a smooth one. In our case we can reduce the dimension of the problem to one, by keeping track of y, the square of the radius of the vector of the price gaps. This reduction is possible because of the quadratic nature of the objective function, and the lack of drift of the uncontrolled price gap. Thus, we trade off high dimension for a non-linearity on the evolution of the system. Our proof strategy is to convert back the one dimensional problem into the original n dimensional problem and check the conditions of the obtained solution using the n-dimensional variational inequality verification theorem for stopping time problems from Øksendal (2000).

As mentioned, Midrigan (2007, 2009) analyzes the effect of monetary policy shock in a general equilibrium model with $n = 2$. He shows that it differs from the one with $n = 1$ because, among other things, the mass of firms that are close to the inaction region just before the shock and find it optimal to adjust right after the shock is smaller when $n = 2$. The reason is that, for $n = 2$, some price adjustments are small because not all individual prices being adjusted are close to the inaction boundary. In the language of Golosov and Lucas (2007), there is a smaller selection effect among the firms that adjust in the $n = 2$ case, compared to the $n = 1$ case. Indeed Caballero and Engel (2007) argue that the cross section distribution of the "desired adjustments", or price gaps in our set up, is one of the key ingredients to understand the aggregate effect in a model with sS policies. Motivated by these findings, we compute the density of the invariant distribution of y, the sum of the square of the price gaps. This density is downward sloping, and, except for its scale, it depends *exclusively* on the number of products n. As n increases this distribution puts relatively more mass on the points close to the boundary of the sphere, i.e. there are more firms close to the point where they want to adjust. As $n \to \infty$, the distribution is uniform.

We characterize the implications for the timing of price changes given $\bar{y}$, σ^2 and n. We show that the expected number of price adjustments per unit of time is given by $n\sigma^2/\bar{y}$, which together with our result for $\bar{y}$ gives a complete characterization of the frequency of price adjustments. This characterization can be used to disentangle the effects on the frequency of adjustments while comparing firms with different number of products, since it points out to all its determinants. Moreover, when used together with other information described below, it can be used to identify the parameters of the model and test its implications. For instance we compare the elasticity of the formulas implied in our paper with the ones implied by tabulations on Bhattarai and Schoenle (2010). We find that the elasticities predicted by the theory are closer to the ones in US data in the case of *constant fixed cost* case.

We solve in closed form for the hazard rate of the price changes as a function of the time elapsed since the last change. The shape of this function, except for its scale, depends *exclusively* on the number of products n. The scale of the function is completely determined by the expected number of adjustment per unit of time, which we have already solve for. For a given n, the hazard rates are increasing in duration, have an elongated S shape, with a finite asymptote. Comparing across different values of n, while keeping the expected number of adjustment constant, we show that the asymptote of the hazard rate is increasing in n. As we let n increase without bound, the asymptote diverges to $+\infty$ and the hazard rate function converges to the one with deterministic adjustments, i.e. towards one with an inverted L shape. In words, as n increases, adjustment is less likely early on, and more likely later on, converging to the extreme case of deterministic adjustment as $n \to \infty$.

Finally we characterize the shape of the distribution of price changes. While price changes occur simultaneously for n products, we characterized the marginal distribution of prices, because this is the object that is usually computed in actual data sets. We give a close form expression for the density of the marginal distribution of price changes as a function of $\bar{y}$ and n. Based on this results we compute several statistics that measure the size of the price changes, such as $\mathbb{E}[|\Delta p|]$, the expected value of the *absolute value* of price changes. We show that, regardless of how the fixed cost changes with n in our two extreme cases, as the number of products increases, the size of the adjustments decreases for all n. Thus, the insight of $n = 2$ generalizes, i.e. with more products the typical adjustment is smaller in each product. We use this statistics, as well as our solution for $\bar{y}$ for different n to compare it with the tabulations in the data from Bhattarai and Schoenle (2010). We find that the elasticities predicted by the theory are close to the ones in the US data for the *constant returns to scale case*.

We show that once the size of the changes is control for, the shape of the price change distribution is *exclusively* a function of the number of products n. We obtain then several statistics that have been computed in the data, such as the coefficient of variation of the absolute value of price changes, or the excess kurtosis, as purely functions of the n. We compare this statistics with the tabulations in US data by Bhattarai and Schoenle (2010) and find the same pattern: higher values of n imply higher dispersion and fatter tails. Indeed the shape of the distribution of price changes is as follows: for $n = 2$ it is bimodal, with modes at the absolute value of $\sqrt{\bar{y}}$, for $n = 3$ is uniform, for $n = 4$ peaks at zero and it is concave, and for larger n it is bell shape. Indeed, as $n \to \infty$, once normalized, the distribution converged to a standard normal. We find the sensitivity of the shape of price changes with respect to n an interesting result to identify different type of models of price adjustments. In particular, bimodality is only predicted for $n = 1$ or $n = 2$. This helps to discriminate with respect to other theories of price adjustments, as the ones bases on a mixture of information and menu cost, worked out in Alvarez, Lippi, and Paciello. Additionally, bimodality receives some support in the data in studies by Cavallo (2010) and Cavallo and Rigobon (2010) which use data from stores that sell large number of products.

2 A stylized multiproduct menu cost model

Let n be the number of goods produced by the firm. Each price p_i evolves according to a random walk without drift, so that $\mathrm{d}p_i = \sigma \; \mathrm{d}W_i$ where $\mathrm{d}W_i$ is a standard Brownian Motion. The n Brownian Motions (BM henceforth) are independent, so $\mathbb{E}\left[W_i(t)W_j(t')\right] = 0$ for all $t, t' \geq 0$ and $i, j = 1, ..., n$. The problem is:

$$V(p) = \min_{\{\tau_j, \Delta p_i(\tau_j)\}_{j=1}^{\infty}} \mathbb{E}\left[\sum_{j=1}^{\infty} e^{-r\tau_j}\psi + \int_0^{\infty} e^{-rt} B\left(\sum_{i=1}^{n} p_i^2(t)\right) dt \,\middle|\, p(0) = p\right] \tag{1}$$

where

$$p_i(t) = \sigma W_i(t) + \sum_{j:\tau_j<t} \Delta p_i(\tau_j) \text{ for all } t \geq 0 \text{ and } i = 1, 2, ..., n, \tag{2}$$

and $p(0) = p$.

So that τ_j are the (stopping) times at which control is exercised. At these times, after paying the cost ψ, the state can be changed to any value in $\mathbb{R}^n$. We denote the vector of price changes as $\Delta p(\tau_j) \in \mathbb{R}^n$. This is a standard adjustment cost problem subject to a fixed cost, with the exception that after paying the adjustment cost ψ the decision maker can adjust the state in the n dimension.

At an abstract level equation (1) and equation (2) can be used to solve a symmetric quadratic loss tracking problem in n dimensions, subject to a fixed adjustment cost. To map it into a tracking problem, let the state of the system be two n dimensional vectors $\hat{p}(t)$, and $p^*(t)$. The interpretation of $\hat{p}(t)$ is the location of the system, and $p^*(t)$ a bliss point, the location that the decision maker is tracking. The instantaneous cost of the decision maker is proportional to the distance between the location of the system and the bliss point, $B||p^*(t) - \hat{p}(t)||^2$, where $B > 0$. Each component of the bliss points evolve as an independent random walk without drift, with variance σ^2 per unit of time. If the decision maker pays a fixed cost ψ/n she can change the location of the system anywhere that she desires. For the purpose of finding the times at which the decision maker chooses to change the state, and to find the value of the changes of the state, we can simplify the problem and consider the distance between the location of the state and the bliss point the of the system, and simple let the state be $p(t) = \hat{p}(t) - p^*(t)$. We have written equation (1) and equation (2) using this "gap" notation.

For future reference we note if B and ψ are multiplied by a constant $\gamma > 0$ the value function is scaled by γ with no change on the decisions. This explains why all the decision are functions of B/ψ. We will use this property to interpret different assumptions about how these parameter vary firms with different number of products n.

We describe now an economic interpretation of the problem, which can be summarized to say that the firms "tracks" the prices that will maximize instantaneous profits from the n products. Consider a system of n independent demands, with constant elasticity η for each product, and a time varying constant marginal cost $C_i(t)$. In the context of the price setting models, our model is a stylized version of the problem introduced by Midrigan (2007, 2009) where the elasticity of substitution between the goods produced *within* the firm is the same as the one of the bundle of goods produced *across* firms. The instantaneous profit maximizing price is proportional to the marginal cost, or in logs $p_i^*(t) = \log C_i(t) + \log((\eta - 1)/\eta)$. In this case we assume that the log of the marginal cost evolve as a random walk with drift so that $p_i^*(t)$ inherits this property. We can interpret the period cost as a second order expansion of the (log) of the profit function with respect to the vector of the log of prices, around the log of the profit maximized price vector. The first order term are zero because we are expanding around $p^*(t)$. The fact that it is an expansion of the *log* of the profit is equivalent to measure the profits relative to the value of the maximized profit for the n goods. There are no second order cross terms due to the separability of the demand. Thus we can write the problem in terms of the gap between the actual price and the profit maximizing price: $p(t) = \hat{p}(t) - p^*(t)$. The constant B is given by $B = (1/2)\eta(\eta - 1)/n$, where the term $1/2$ is due to a second order expansion, the terms with η are due to the fact that the curvature of the profits depend on the elasticity of demand, and the term $(1/n)$ is the the share of profits from each product relative to the profits across the n goods. In this interpretation the value of the fixed cost is measured relative to the profit of the n goods, thus it costs ψ/n in units of the numeraire good. Since all that matter for the decision is the ratio of B to ψ this normalization only scales the units of the vale function. In Appendix D we give a derivation of the second order expansion as well as a discussion and interpretation of the parameters B and ψ. In particular the interpretation of how to scale B and ψ for firms with different of products.

Below we consider two cases for scaling of the cost of adjustment with respect to the number of goods. In the first case, which we refer to as constant returns to scale (CRTS) technology for adjustment cost, when we compare firms with different values of n, the adjustment cost scales linearly with it, so that $\psi = n\ \psi_1$. In this case a firm with twice as many products pays twice as much in terms of the numeraire good to adjust all the prices simultaneously. We refer to the case of a *constant fixed cost*, if $\psi = \psi_1$, so that a firm with twice as many products pays the same cost in terms of numeraire to adjust the twice as many prices. We think that these two extreme simple cases bracket all of the interesting setups.

We note the following basic properties of the value function and the optimal policy.

1. Given the symmetry of the BM and of the objective function around zero, and the independence of the BM's, one can use reflection around zero to show that the value function only depends on the absolute values of p_i, i.e. $V(p) = V(|p_1|, |p_2|, ..., |p_n|)$ for all $p \in \mathbb{R}^n$.

2. Due to the the symmetry of the return function, in the law of motion the target prices and the lack of drift, it is easy to see that after an adjustment the state is reset at the origin, i.e. $p(\tau_j^+) = 0$, or $\Delta p(\tau_j) = -p(\tau_j^-)$. See Appendix A for a formal argument.

3. The state space $\mathbb{R}^n$ can be divided in two regions, an inaction region $\mathcal{I} \subset \mathbb{R}^n$ and control region $\mathcal{C} \subset \mathbb{R}^n$. We use $\text{Int}(\mathcal{C})$ for the interior of the control region and $\partial\mathcal{I}$ for the boundary of the inaction region. We have that $\mathcal{C} \cap \mathcal{I} = \emptyset$, that inaction is strictly preferred in $\mathcal{I}$, that control is strictly preferred in $\text{Int}(\mathcal{C})$, and that in $\partial\mathcal{I}$ the agent is

indifferent between control and inaction.

We write down the conditions for the solution of the problem, provided that a value function is smooth enough, i.e. we look for a solution of the "strong" formulation of the problem with: $V \in C^1(\mathbb{R}^n)$ and $V \in C^2(\mathbb{R}^n \backslash \partial\mathcal{I})$, so the function is once differentiable in the whole domain, and twice differentiable everywhere, but in the boundary of the inaction set. In the range of inaction the cost for the firm is given by the following Bellman equation:

$$r\ V(p_1, p_2, ..., p_n) = B\ \sum_{i=1}^{n} p_i^2 + \frac{\sigma^2}{2} \sum_{i=1}^{n} V_{ii}(p_1, p_2, ..., p_n) \tag{3}$$

for all $p \in \mathcal{I}$. In the control region we have:

$$V(p_1, p_2, ..., p_n) = V(0) + \psi \tag{4}$$

for all $p \in \mathcal{C}$. The optimality of returning to the origin implies that,

$$V_i(0, 0, ..., 0) = 0 \text{ for all } i = 1, 2, ..., n. \tag{5}$$

Finally, differentiability in the boundary of the inaction region gives

$$V_i(p_1, p_2, ..., p_n) = 0 \text{ for } i = 1, 2, ..., n \text{ and for all } p \in \partial\mathcal{I}, \tag{6}$$

We refer to this condition as smooth pasting.

We briefly comments on the results on control theory that apply to our problem. Theorem 1 in Baccarin (2009) shows the existence of a continuous value function V and a policy described by a continuation and control region for a class of problem that include ours. The set-up in Baccarin (2009) includes a more general form of adjustment cost, more general period return function, and more general law of motion for the state, as well as weaker differentiability assumption on these function.[5] Øksendal (2000) and Aliev (2007) analyze

[5] Strictly speaking, our problem does not fit one of the assumptions for Theorem 1 in Baccarin (2009).

a general class of slightly simpler stopping time problem in n dimensions. Their consider a problem with a one time decision of when to collect a given reward function of the state, denoted by g. Before that time the decision maker has either zero flow returns, or in the case of Øksendal (2000) she receives a flow return f, as function the state. The decision maker maximized the expected discounted value of the reward.[6] Their problem maps into our by making the reward $g(p) = V(0) + \psi$ and the flow return $f(p) = B||p||^2$. Aliev (2007) shows that equation (6) is necessary for optimality, provided that $p \in \partial\mathcal{I}$ is a regular point for the stopping set $\mathcal{C}$ with respect to the process $\{\sigma W(t)\}$ and that the derivatives of the value function in a neighborhood of $\partial\mathcal{I}$ are bounded. Theorem 10.4.1 in Øksendal (2000) is a verification theorem in term of variational inequalities which, when adapted to our set-up, says that if a function V that satisfies conditions equations (3)-(6) and several additional conditions -which we state and check in our proof- the value function solves the stopping time problem.

3 Characterization of the solution

Before presenting the solution of this problem we change the state space, which we summarize using a single variable. Let

$$y = \sum_{i=1}^{n} p_i^2 \tag{7}$$

measure the deviation of prices from their optimal value across the n goods. We consider policies summarized by a single number $\bar{y}$. In this class of policies the firm controls the state so that if $y < \bar{y}$, there is inaction. The first time that y reaches $\bar{y}$, all prices are adjusted to the origin, so that $y = 0$. We will find the optimal policy in this class. Then we will show that the optimal policy of the original problem is of this form.

In particular, Assumption (2.4) requires that the cost diverges to infinity as the norm of the adjustment diverges. Nevertheless, we can artificially modify our problem by incorporating a proportional adjustment cost that applies only when $||p||$ is very large, without altering our solution.

[6] While in their analysis discounting is not explicitly included, it is easy to introduce it by taking time as one of the n states.

The variable y measures the square of the ray of a sphere centered on the origin. Since each of the prices follows identical independent standard BM in the inaction region, then y follows a simple diffusion in the inaction. Using Ito's Lemma on equation (7) the evolution of y is

$$\mathrm{d}y = n\sigma^2 \mathrm{d}t + 2\sigma \sum_{i=1}^{n} p_i(t)\mathrm{d}W_i$$

This implies that the quadratic variation of y is:

$$\mathbb{E}(\mathrm{d}y)^2 = 4\sigma^2 \left(\sum_{i=1}^{n} p_i^2(t) \right) \mathrm{d}t$$

Thus we can define a stochastic differential equation for y with a new standard BM $\{W(t)\}$ that solves:

$$\mathrm{d}y = n\sigma^2 \mathrm{d}t + 2\sigma\sqrt{y}\ \mathrm{d}W \text{ for } y \in [0, \bar{y}]. \tag{8}$$

We note that for the unregulated process, i.e. when $\bar{y} = \infty$, if $y(0) > 0$ then $y(t) > 0$ for $t > 0$ with probability one provided that $n \geq 2$, see Karatzas and Shreve (1991) Proposition 3.22.[7]

Note that the drift and diffusion terms in equation (8) are only functions of y. We also note that instantaneous return is a function of y, so we can write the following

$$v(y) = \min_{\bar{y}} \mathbb{E}\left[\sum_{j=1}^{\infty} e^{-r\tau_j}\psi + \int_0^{\infty} e^{-rt} B\ y(t)\ dt \ \middle|\ y(0) = y \right] \tag{9}$$

subject to equation (8) when $y \in [0, \bar{y}]$, where τ_j are the first time that $y(t)$ hits $\bar{y}$. The function v solves:

$$r\ v(y) = By + n\sigma^2\ v'(y) + 2\sigma^2 y\ v''(y), \quad \text{ for } y \in (0, \bar{y})\ . \tag{10}$$

[7]This result was obtained for Bessel processes, which are the square root of $y(t)$. Additionally, Karatzas and Shreve (1991) have shown in Problem 3.23 and 3.24 that for if $y(0) > 0$, then for $n = 2$ the unregulated process can become arbitrarily close to zero but for $n \geq 3$ almost every path remains bounded away from zero. Furthermore, for the regulated process the classification for the boundaries of a diffusion gives that for $n \geq 2$ the point $y = 0$ is an entrance boundary, as verified in Karlin and Taylor (1999) Example 6, Chapter 12.6.

Since policy calls for adjustment at values higher than $\bar{y}$ we have:

$$v(y) = v(0) + \psi, \qquad \text{for all } y \geq \bar{y} \; . \tag{11}$$

If v is differentiable at $\bar{y}$ we can write the two boundary conditions:

$$v(\bar{y}) = v(0) + \psi \quad \text{and} \quad v'(\bar{y}) = 0 \; . \tag{12}$$

These conditions are typically referred to as value matching and smooth pasting. For $y = 0$ to be the optimal return point, it must be a global minimum, and thus we require that:

$$v'(0) \geq 0 \; . \tag{13}$$

Note the weak inequality, since y is non-negative.

The next proposition finds an analytical solution for v in the range of inaction.

Proposition 1. Let $\sigma > 0$. The ODE given by equation (10) is solved by the following analytical function:

$$v(y) = \sum_{i=0}^{\infty} \beta_i \, y^i \; , \qquad \text{for } y \in [0, \bar{y}] \; . \tag{14}$$

where the coefficients $\{\beta_i\}$ solve:

$$\beta_0 = \frac{n\sigma^2}{r}\beta_1 \quad , \quad \beta_2 = \frac{r\beta_1 - B}{2\sigma^2(n+2)} \; , \quad \beta_{i+1} = \frac{r}{(i+1)\; \sigma^2 \; (n+2i)} \, \beta_i \; , \qquad \text{for } i \geq 2 \; . \tag{15}$$

for any β_0.

The proof follows by replacing the function given in equation (14) into the ODE (10) and matching the coefficients for the powers of y^i. By the Cauchy-Hadamard theorem, the power series converges absolutely for all $y > 0$ since $\lim_{i \to \infty} \beta_{i+1}/\beta_i = 0$. The next proposition shows that there exist a unique solution of the ODE (10) satisfying the relevant boundary conditions. It partially characterizes the solution of the threshold $\bar{y}$.

PROPOSITION 2. Assume $r > 0, \sigma > 0, n \geq 1$. There exists $\bar{y}$ and a unique solution of the ODE (10) satisfying the two boundary conditions described in equations (12) for which $v(\cdot)$ satisfies: i) it is minimized at $y = 0$, ii) it is strictly increasing in $(0, \bar{y})$, and iii) $\bar{y}$ is a local maximum, i.e. $v''(\bar{y}) < 0$. The unique value of $\bar{y}(\psi)$ is strictly increasing in ψ, with $\bar{y}(0) = 0$ and $\bar{y} \to \infty$ as $\psi \to \infty$. Moreover, as $r \downarrow 0$, the optimal threshold satisfies $\bar{y} = \sigma^2\, Q\left(\frac{\psi}{B\,\sigma^2},\ n\right)$ for some strictly increasing function $Q(\cdot, n)$.

We note that the solution for $\bar{y}$ is only a function of the ratio ψ/B is apparent from the definition of the sequence problem. That it is strictly increasing in the ratio of the fixed cost to the benefit of adjustment ψ/B is quite intuitive. The last property implies that if $\bar{y}$ is the solution of the problem for parameters $\sigma, \psi/B$ and n, then $\bar{y}' = \bar{y}\,\gamma$ is the solution for parameters $(\psi/B)' = \gamma$, $n' = n$ and $\sigma' = \sqrt{\gamma}\,\sigma$ for any $\gamma > 0$.

While the last proposition gives a partial characterization of the optimal thresholds, it does not analyze how it depends on n. To do so, we give a simple expression for an approximate solution of $\bar{y}$. The formula gives an approximate solution for small values of ψ and r. Mechanically we take value matching and smooth pasting conditions derived above and we set $\beta_i = 0$ for $i \geq 3$, i.e. we assume that $v(\cdot)$ is quadratic (see Appendix B). Taking r to zero gives:

$$\bar{y} = \sqrt{\frac{\psi\,\sigma^2\,2\,(n+2)}{B}}\ . \tag{16}$$

Notice that this approximation satisfies the properties of the general solution of Proposition 2. The effect on this formula of $\psi\sigma^2/B$ is exactly the same as in the case of one product. Indeed the quartic root implied when $n = 1$ was obtained by Dixit (1991) for the model with $n = 1$. We found that the quadratic approximation to $v(\cdot)$, which amounts to a quartic approximation to $V(\cdot)$, gives very accurate values for $\bar{y}$. Alternatively, the rest of the β_i coefficients become very small for $i \geq 3$ for the parameters that we are interested (see Appendix C for documentation).

The next proposition verifies that the restricted policy that we use to characterize the solution is indeed optimal. Indeed the decision rule is very similar to the one obtained by Baccarin (2009). He studies a general class of problems with fixed and variable adjustment cost and a non-linear cost function, but illustrate his results with the computation of several examples, among them one very similar to ours.

Proposition 3. Let v be the solution of the restricted problem equation (9) and equation (8). Let $V(p) = v\left(\sum_{i=1}^{n} p_i^2\right)$. This is the solution of the problem described equation (1) and equation (2).

For completeness we note that the in the case of $n = 1$ and the case of perfectly correlated target prices. In the case of one product, i.e. $n = 1$, the solution to V is easily seen to be the sum of a quadratic and an two exponential, as in

$$V(p) = \frac{B}{r}p^2 + \beta\left(\exp\left(\zeta p\right) + \exp\left(-\zeta p\right)\right) + B\left(\frac{\sigma}{r}\right)^2$$

where $\zeta = \sqrt{2r}/\sigma$ and the constant β is chosen to match enforce smooth pasting and value matching. Moreover, it is easy to see that in that case the $v(y) = V(\sqrt{y})$ solves the ODE in (10) and its boundary conditions. We note that the solution for the $n = 1$ case and the expression for the approximation for $\bar{y}$ are the same as the ones derived in Dixit (1991), which we explore in the price setting context in Alvarez, Lippi, and Paciello. In the case of n *perfectly correlated target prices* the problem has, after the first adjustment, a single state variable. In this case, in terms of the threshold policy and value function, the problem is identical to the one with only one price. The static return is thus $nBp(t)^2$ where $p(t)$ is, when uncontrolled, a one dimensional brownian motion. The only difference with the problem with only one price is that the value of B is multiplied by n, or more importantly, the ratio B/ψ is proportional to n. This is quite natural, since the adjustment has the same effectiveness for all products, and hence it is as if it were cheaper. Note that, in the case of the CRTS assumption, the value of the adjustment threshold, and hence the frequency of adjustment, is independent of n. Instead, in terms of the implication for price changes, the problem with perfectly correlated shocks is quite differently, since there are *no* small price changes. When adjustment takes place, all products have the same price gap. We return to this simple case later on to speculate on the case of positive, but less than one correlation between the innovations.

4 Implications for timing and size of price changes

In this section we explore the implications for the frequency and distribution of price changes.

We let the expected time for $y(t)$ to hit the barrier $\bar{y}$ starting at y by the function $\mathcal{T}(y)$. This function satisfies:

$$0 = 1 + \ n\,\sigma^2\,\mathcal{T}'(y) + \ 2\,y\,\sigma^2\,\mathcal{T}''(y) \ \text{ for } y \in (0, \bar{y}) \quad \text{and} \quad \mathcal{T}(\bar{y}) = 0 \ ,$$

where the first condition gives the law of motion inside the range of inaction and the second one imposes the terminal condition on the boundary of the range of inaction. The unique solution of this ODE that satisfies the relevant boundary condition is:

$$\mathcal{T}(y) = \frac{\bar{y} - y}{n\,\sigma^2} \ \text{ for } y \in [0, \bar{y}] \ . \tag{17}$$

We use $\mathcal{T}(0)$ as the expected time between successive price adjustments, and thus the average number of adjustment, denoted by N_a is given by $\frac{1}{\mathcal{T}(0)}$. We collect this result in the following proposition:

Proposition 4. Let N_a be the expected number of price changes for a multi-product firm with n goods. It is given by

$$N_a = \frac{n\sigma^2}{\bar{y}} = \frac{n}{Q\left(\frac{\psi}{B\ \sigma^2},\ n\right)} \cong \sqrt{\frac{B\ \sigma^2}{2\ \psi}\frac{n^2}{(n+2)}} \ . \tag{18}$$

In the first equality of expression of equation (18) we use the function $Q(\cdot)$ derived in Proposition 2 and in the last one we use the approximation of $\bar{y}$ for small ψ and r (see Appendix C for more documentation on the accuracy of the approximation). It is interesting that this expression extends the well known expression for the case of $n = 1$, simply by adjusting the value of the variance from σ^2 to $n\sigma^2$. The number of products n affects N_a through two opposing forces. One is that with more products, the variance of the deviations of the price gaps increases, and thus a given value of $\bar{y}$ is hit sooner in expected value, which we refer to as the direct effect. On the other hand, with more products, the optimal value of $\bar{y}$ is higher. Expression equation (18) shows that, as often happens in these models, the direct effect dominates, and the frequency of adjustment increases with n.

We use this expression to study how the bundling of menu costs, i.e. the fact that a single menu cost relates to several products, affects the frequency of adjustment of individual prices. This is interesting because recent evidence in Bhattarai and Schoenle (2010) shows that the the frequency of price adjustment appears higher for firms that sell a larger number of goods.[8] They find that the average frequency of price adjustment increases.

Table 1: Frequency of Price changes N_a

	number of products n						implied
	1	2	4	6	10	50	ψ_1
N_a for C.R.T.S. model $\psi = n\ \psi_1$:	2	2.4	2.8	2.9	3.1	3.3	0.02
N_a for C.F.C. model: $\psi = \psi_1$:	1.4	2.4	3.9	5.1	6.9	17	0.04
N_a for US Data:	-	2.4	2.3	2.8	3.5	-	-

Value of $\frac{\psi_1}{B\sigma^2}$ chosen to match the size of price changes at $n = 2$. US data from Bhattarai and Schoenle (2010) Figure 1. Implied ψ_1 using $B = 20$ and $\sigma = 0.15$.

[8] See Figures 1 and 2 in their paper. These authors group firms into 4 bins, according to the number of items sold (and recorded by the BLS), from 1 to 3 goods in the first bin to more than 7 goods in the fourth bin. They first measure the frequency of price changes at the good level, then compute the median frequency across the goods produced in the firm. Finally, they average these medians inside each of the 4 bins.

This pattern is qualitatively consistent with the formula in equation (18), which shows that N_a is increasing in n. Notice however that in this comparison we are keeping ψ constant, so that as n increases the menu cost per good is decreasing. One may wonder whether the increased activity by the firms follows from the fact that the menu cost is smaller (per good) or because of the bundling of the goods prices. To separate the effects of the economies of scale in the menu cost from the bundling of the goods, consider the case where the cost ψ grows linearly with the number of goods n, i.e. : $\psi = \psi_1 \, n$. This gives

$$N_a \cong \sqrt{\frac{B \; \sigma^2}{2 \; \psi_1} \frac{n}{(n+2)}} \; . \tag{19}$$

which is also increasing in n, although at a lower rate. Thus, even under "CRTS" for the menu cost, the bundling of the goods pricing induces more frequent adjustments than in the case where the menu costs are dissociated, i.e. when $n = 1$. Table 1 uses equation (19) for the case of constant returns to scale (CRTS) and equation (18) for the case of constant fixed cost (CFC) to calculate hypothetical values of N_a for different values of n. For both cases we have selected the values of $B \; \sigma^2/\psi_1$ so that its value is 2.4 adjustments per year, the value estimated by Bhattarai and Schoenle (2010) for the US for firms with $n = 2$.

Table 1 also includes a column with US data. Comparing the case of CRTS with the one with CFC, the latter displays a pattern much closer to the one in the US data.

We now study the invariant distribution of the sum of the squares of the price gaps $||p||^2 = \sum_{i=1}^{n} p_i(t)^2$ under the optimal policy. We will denote the density of the invariant distribution by $f(y)$ for $y \in [0, \bar{y}]$. This is interesting to study the response of a set of firms that are in steady state (i.e. in the invariant distribution) to an unexpected shock to their target which will displace the price gaps uniformly. In particular the study of how much mass is close to the boundary of inaction (so that after the "unexpected shock" they will decide to adjust) is one that has been identify as one of the key determinants to the impact effect of monetary policy shocks. We are interested in studying how this mass changes as we vary the number of goods n. There is an extensive literature on this topic, see for example Caballero and Engel (2007). The density of the invariant distribution has to solve the forward Kolmogorov equation and relevant boundary conditions.

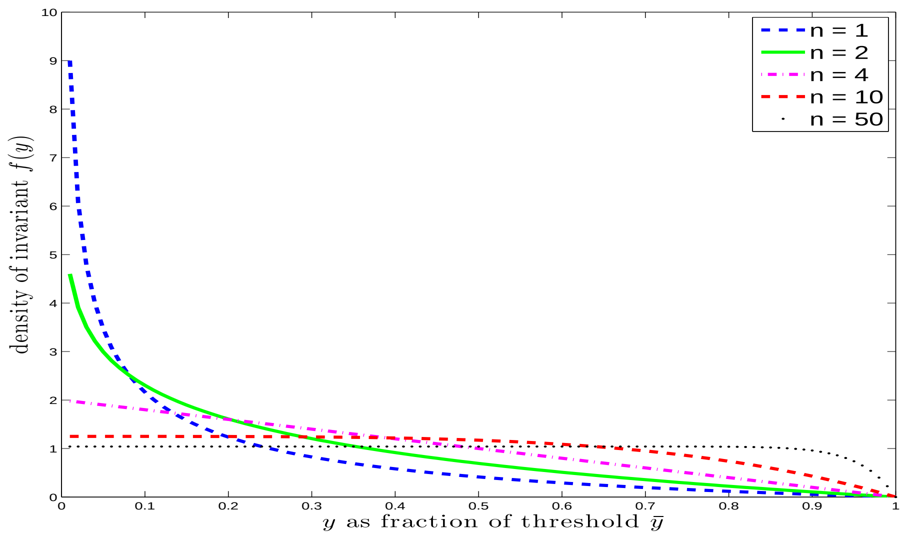
PROPOSITION 5. The density $f(\cdot)$ of the invariant distribution of the sum of the squares of the price gaps y, for a given thresholds $\bar{y}$ in the case of $n \geq 1$ products is for all $y \in [0, \bar{y}]$

$$
\begin{aligned}
f(y) &= \frac{1}{\bar{y}} \left[\log(\bar{y}) - \log(y)\right] \text{ if } n = 2, \text{ and} \\
f(y) &= (\bar{y})^{-\frac{n}{2}} \left(\frac{n}{n-2}\right) \left[(\bar{y})^{\frac{n}{2}-1} - (y)^{\frac{n}{2}-1}\right] \text{ otherwise.}
\end{aligned} \tag{20}
$$

So the density has a peak at $y = 0$, decreases in y, and reaches zero at $\bar{y}$. The shape depends on n. The density is convex in y for $n = 1, 2$ and $n = 3$, linear for $n = 4$, and concave for $n \geq 5$. This is intuitive, since the drift of the process for y increases linearly with n, hence the mass accumulates closer to the upper bound $\bar{y}$ for higher n. Indeed as $n \to \infty$ the distribution converges to a uniform in $[0, \bar{y}]$. Proposition 5 makes clear also that the shape of the invariant density depends exclusively on n, the value of the other parameters, ψ, B, σ^2 only enters as determining $\bar{y}$, which only stretches the horizontal axis proportionally.

Consider the effect of decreasing the vector of price gaps by a constant $\delta > 0$ in all dimensions. The interpretation of this experiment, is the effect of an unexpected jump in the target price for all the goods. We want to find out the fraction of firms under the invariant that will adjust their prices. We will assume that from here on the price gap process remains the same, so the firms solved the problem stated above. For this we need the invariant distribution of p, not just y. We note that for each $y \leq \bar{y}$, the distribution of p is uniform on the n-dimensional sphere with radius $\sqrt{y}$. This is due to the symmetry of the distributions of the price gaps in each dimension, and its independence. Following the notation in Song and Gupta (1997) we use $U(n, 2)$ for the uniform distribution of p on the n-dimensional sphere, i.e. all the values of p with $||p|| = 1$ have the same density. Now we obtain the fraction of prices p with $||p||^2 = y \leq \bar{y}$ that after the δ "shock" will be outside the set of inaction. They

Figure 1: $f(\cdot)$ density of invariant distribution of y, for various choices of n



are given by:

$$||p||^2 - 2\delta \left(\sum_{i=1}^{n} p_i \right) + n\delta^2 > \bar{y} \quad \text{or} \quad \frac{\sum_{i=1}^{n} p_i}{\sqrt{y}} \leq \frac{y - \bar{y}}{2\delta\sqrt{y}} + n\frac{\delta}{\sqrt{y}} \ . \tag{21}$$

Abusing notation let $u(x; n, 2)$ be the density of $U(n, 2)$ and define:

$$D(s) = \int_{\{x:\ ||x||=1\}} \mathbf{I} \left\{ \sum_{i=1}^{n} p_i \leq s \right\} \ u(x_1, ..., x_n; n, 2) \ dx_1 \ dx_2 \cdots dx_n \ .$$

Thus the fraction of firms that adjust after the unexpected increase in prices δ, denoted by is given by $A(\delta)$:

$$A(\delta) = \int_0^{\bar{y}} f(y) \ D \left(\frac{y - \bar{y}}{2\delta\sqrt{y}} + n\frac{\delta}{\sqrt{y}} \right) dy \approx -f'(\bar{y}) D \left(n\frac{\delta}{\sqrt{\bar{y}}} \right) \frac{\bar{y}^2}{2} = \frac{n}{2} D \left(n\frac{\delta}{\sqrt{\bar{y}}} \right) \ , \tag{22}$$

where the last term is a an approximation of $A(\delta)$ using a first order expansion of the product fD around $\bar{y}$, which will be accurate for small values of $\bar{y}$.

We now move to the study of the hazard rate of price adjustments.

Proposition 6. Let t denote the time elapsed since the last price change. Let $J_\nu(\cdot)$ be the Bessel function of the first kind. The hazard rate for price changes is given by

$$h(t) = \sum_{k=1}^{\infty} \frac{\xi_{n,k}}{\sum_{s=1}^{\infty} \xi_{n,s} \exp\left(-\frac{q_{n,s}^2 \sigma^2}{2\bar{y}}\, t\right)} \frac{q_{n,k}^2 \sigma^2}{2\bar{y}} \exp\left(-\frac{q_{n,k}^2 \sigma^2}{2\bar{y}}\, t\right) \;, \text{ where } \nu = \frac{n}{2} - 1 \;,$$

$$\xi_{n,k} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{q_{n,k}^{\nu-1}}{J_{\nu+1}(q_{n,k})} \;, \quad \text{and } q_{nk} \text{ are the positive zeros of } J_\nu(\cdot),$$

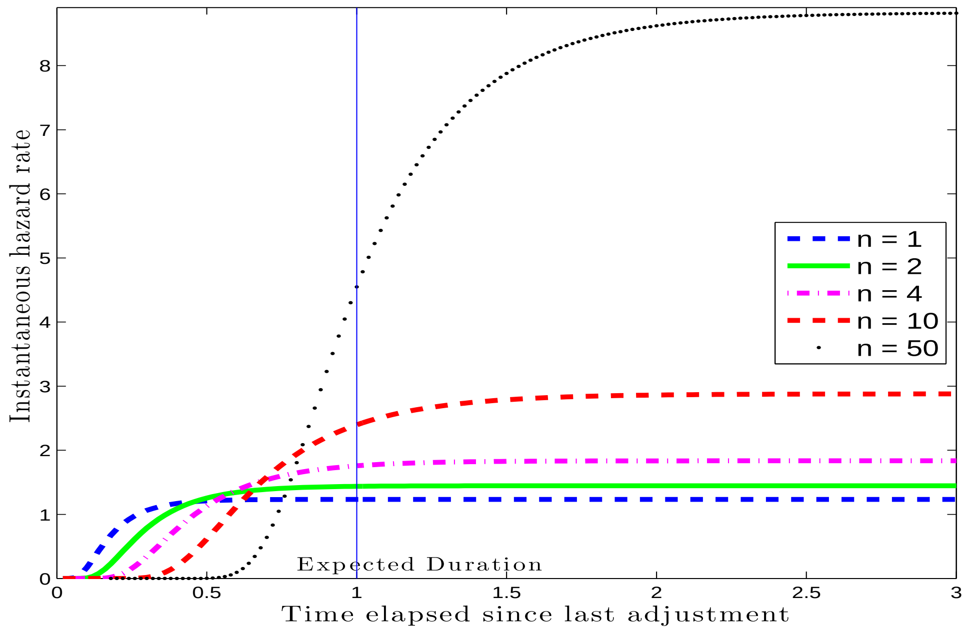
$$\text{which asymptotes to } \lim_{t\to\infty} \frac{h_n(t)}{\mathcal{T}(0)} = \frac{q_{n,1}^2}{2n} > \frac{(n-1)^2}{2n} \;. \tag{23}$$

In the proof we use results from probability theory on the first passage of time of a n brownian motion in a sphere center to the origin by Ciesielski and Taylor (1962) as well as characterization from the zeros of the Bessel function from Qu and Wong (1999) and Hethcote (1970).

Proposition 6 compares the asymptote of the hazard rate with the expected time until adjustment, which equals $\mathcal{T}(0) = \bar{y}/(n\sigma^2)$, as derived above. Notice that for a model with constant hazard rate these two quantities are the reciprocal of each other, i.e. the expected duration is the reciprocal of the hazard rate. We use this ratio, as a function of n as a measure of how close the model is to have constant hazard rates. We note that this ratio is exclusively a function of n. Indeed from the expression Proposition 6, it is immediate that the shape of the hazard rate function depends only on the number of products n. Changes in σ^2, B, ψ only stretch linearly the horizontal axis. More precisely, once keeping the expected time until adjustment $\mathcal{T}(0)$ fixed, the hazard rate is only a function of n.

Figure 2 plots the hazard rate function h for different choices of n keeping the expected time between price adjustment fixed at one. As Proposition 6 shows the function h has an asymptote, which is increasing in the number of products n. Moreover, since the asymptote diverges to ∞ as n increases with no bound, the hazard rate converges to a an inverted L shape, as the one for a model where adjustment are done exactly every $\mathcal{T}(0) = 1$ periods. To see this note that, defining $\tilde{y} \equiv y/\bar{y}$ and fixing the ratio $\sigma^2/\bar{y} = \mathcal{T}(0)/n$ so that for any n

Figure 2: Hazard rate of Price Adjustments for various choices of n



For each n the value of $\sigma^2/\bar{y}$ is chosen so that the expected time elapsed between adjustments is one.

the expected time elapsed between price changes is $\mathcal{T}(0)$, we have:

$$d\tilde{y} = \mathcal{T}(0)\mathrm{d}t + 2\sqrt{\tilde{y}\,\frac{\mathcal{T}(0)}{n}}\,\mathrm{d}W \text{ for } \tilde{y} \in [0,1]. \tag{24}$$

As $n \to \infty$ the process for the normalized size of the price gap $\tilde{y}$ described in equation (24) converges to the deterministic one, in which case the hazard rate is zero between times 0 and below $\mathcal{T}(0)$ and ∞ precisely at $\mathcal{T}(0)$. For completeness, Table 2 computes the first zero for the relevant Bessel functions and the asymptotic hazard rate for several value of n.

The shape of estimated hazard rates varies across studies, but many have found flat or decreasing ones, and some have found hump-shape ones. As can be seen from Figure 2 the hazard rate for the case of $n = 1$ is increasing but rapidly reaches its asymptote. As n is increased, the shape of the hazard rate becomes closer to the inverted L shape of its limit as $n \to \infty$. For instance, when $n = 10$ the level of the hazard rate evaluated at the expected duration is about twice as large as the one for $n = 2$. This is a prediction that can be tested

Table 2: Limit hazard rates for various values of n

	number of products n									
	1	2	3	4	6	8	10	20	50	100
zeroes of $J_{\frac{n}{2}-1}(\cdot) : q_{n1}$	1.6.	2.4	3.1	3.8	5.1	6.4	7.6	13	30	56
$\frac{\text{Limit Hazard rate}}{\text{Expected duration}} : \lim_{t\to\infty} \frac{h_n(t,\bar{y})}{\mathcal{T}(0)}$	1.2	1.4	1.6	1.8	2.2	2.5	2.9	4.5	8.8	16

Note: for $n = 1$ and $n = 3$ the zeros are multiples of π, i.e. $q_{1,k} = (2(k-1)+1)\pi/2$ and $q_{3,k} = k\,\pi$.

using the data set in Bhattarai and Schoenle (2010).

Finally we discuss the distribution of price changes. This distribution can be described in terms of 2 parameters: the number of goods n, and the optimal threshold describing the size of the inaction set $\bar{y}$. The value of $\bar{y}$, as discussed above, depends on all the parameters. Since after an adjustment price gaps are set to zero, price changes coincide with the value of $p(\tau) \in \partial\mathcal{I} \subset \mathbb{R}^n$, the surface of an n-dimensional sphere of radius $\sqrt{\bar{y}}$. Let τ, be a time where p hits he boundary of the range of inaction, then given that each of the $p_i(t)$ when they are uncontrolled, are independently and identically normally distributed, price changes $\Delta p(\tau) = -p(\tau)$ are uniformly distributed in the n-dimensional surface of the sphere of radius $\sqrt{\bar{y}}$ [9]. The next proposition characterizes the marginal distribution of price changes.

Proposition 7. Let $\Delta p \in \partial\mathcal{I} \subset \mathbb{R}^n$ denote a price change for the n goods. The distribution of the price change of an individual good, i.e. the marginal distribution of $\Delta p_i \in [0,\ \sqrt{\bar{y}}\ ]$ has density:

$$w(\Delta p_i) = \frac{1}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\ \sqrt{\bar{y}}} \left(1 - \left(\frac{\Delta p_i}{\sqrt{\bar{y}}}\right)^2\right)^{(n-3)/2} \tag{25}$$

where $Beta(\cdot,\cdot)$ denotes the Beta function. The standard deviation and kurtosis of the price

[9]The distribution of $\Delta p(\tau)$ is uniform in the surface of the sphere since the p.d.f. of a jointly normally distributed vector of n identical and independent normals, apart from a constant, is given by the exponential of a the square of an sphere with radius $\sqrt{\bar{y}}$.

changes, and expected value of the absolute value of price changes and its coefficient of variations are given by:

$$\begin{aligned} Std(\ \Delta p_i\ ) &= \sqrt{\bar{y}/n}\ , \qquad Kurt(\ \Delta p_i\ ) = \frac{3\ n}{n+2}\ , \\ \mathbb{E}[\ |\Delta p_i|\ ] &= \frac{\sqrt{\bar{y}}}{\frac{n-1}{2}\ Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\ , \ \frac{Std(\ |\Delta p_i|\ )}{\mathbb{E}(\ |\Delta p_i|\ )} = \sqrt{\left[\frac{n-1}{2}Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\right]^2\frac{1}{n}-1}\ . \end{aligned}$$

Moreover, as $n \to \infty$, the distribution of $\Delta p_i/Std(\Delta p_i)$ converges point-wise to a standard normal.

Using the previous proposition and the approximation for $\bar{y}$ we obtain the following expression for the standard deviation of price changes:

$$Std(\ \Delta p_i\ ) = \left(\frac{\sigma^2\psi}{B}\frac{2(n+2)}{n^2}\right)^{1/4} \text{ and in the CRTS case } Std(\ \Delta p_i\ ) = \left(\frac{\sigma^2\psi_1}{B}\frac{2(n+2)}{n}\right)^{1/4},$$

where both expressions are decreasing in n. The expression for the kurtosis of the price changes shows that this statistic is an increasing function of n.

We can approximate some of the expressions in Proposition 7 for statistics for $|\Delta p_i|$ involving the *Beta* function to obtain the following simpler expressions:[10]

$$\begin{aligned} \mathbb{E}[\ |\Delta p_i|\ ] &\approx \sqrt{\bar{y}/n}\ \sqrt{\frac{2}{\pi}}\ \sqrt{1+\frac{1.1}{2\ n}} = Std(\ \Delta p_i\ )\ \sqrt{\frac{2}{\pi}}\ \sqrt{1+\frac{1.1}{2\ n}} \quad \text{and} \\ \frac{Std(\ |\Delta p_i|\ )}{\mathbb{E}(\ |\Delta p_i|\ )} &\approx \sqrt{\frac{\pi}{2}\left(\frac{2n}{1.1+2n}\right)-1}\ . \end{aligned}$$

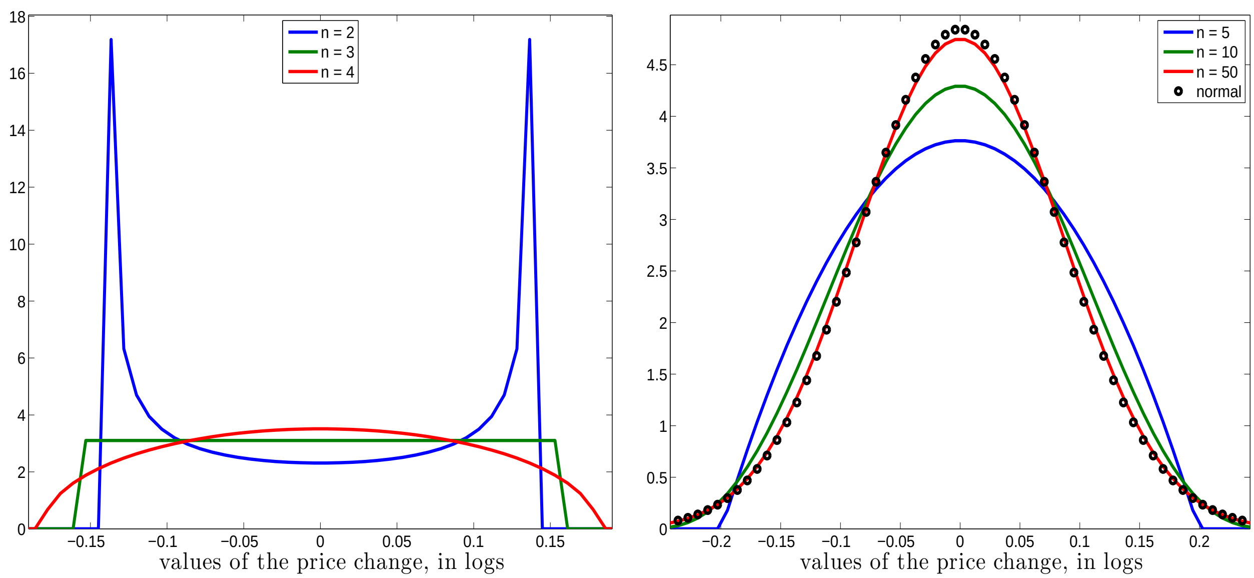
The expression for the approximate value of $\mathbb{E}[|\Delta p_i|]$ is given by $Std(\Delta p_i)$ times a decreasing function of n. The expression for the approximate value of $Std(|\Delta p_i|)/\mathbb{E}(|\Delta p_i|)$ show that this statistic is an increasing function of n.

We note that the shape of the distribution h for price changes differs substantially for small values of n. For $n = 2$ is U-shaped, for $n = 3$ is uniform, for $n = 4$ it has the shape of

[10] We note that error on the approximation error for $\mathbb{E}[\ |\Delta p_i|\ ]$ and $Std(\ |\Delta p_i|\ )/\mathbb{E}(\ |\Delta p_i|\ )$ are smaller than 0.26% and 0.91%.

a half circle, and for $n \geq 6$ it has bell shape.[11] Proposition 7 establishes that when $n \to \infty$ the distribution converges to a normal: this can be seen in Figure 3 by the comparison of the distribution for $n = 50$ and the p.d.d. of a normal distribution with standard deviation equal to $Std(\Delta p_i)$ for $n = 50$.

Figure 3: Density $w(\cdot)$ of the price changes for various choices of n



Parameter values: $B = 20$, $\sigma = 0.15$, $\psi_1 = 0.02$. Menu cost proportional to n. Solid lines are the p.d.f for w for different n. Circles denote the p.d.f. of a normal with standard deviation equal to that of Δp_i for $n = 50$.

Table 3 computes the size of the price adjustments, measured as $E[|\Delta p|]$, as a function of n. We do so for the two extreme technologies, the constant returns to scale (CRTS) and the constant fixed cost (CFC) case. In each case we fix the value of the parameter $B\sigma^2/\psi_1$ so that this statistic is 0.085, the value estimated by Bhattarai and Schoenle (2010) in US data. We also report the values estimated for the US for other values of n. Comparing both assumptions, it seems that the US data is somewhere in the middle, but closer to the case of CRTS.

Furthermore, from the expressions in Proposition 7 the distribution of price changes Δp,

[11]For n equal to 2, 3 and 4 one can grasp the shape of the distribution h from geometrical considerations, together with the fact the maximum of a density of a univariate normal is at one.

Table 3: Size of Price changes $E[|\Delta p|]$

	number of products n						implied
	1	2	4	6	10	50	ψ_1
$E[|\Delta p|]$ for C.R.T.S. model $\psi = \; n \; \psi_1$:	10%	8.5%	7.5%	7.1%	6.8%	6.4%	0.03
$E[|\Delta p|]$ for C.F.C. model: $\psi = \psi_1$:	12%	8.5%	6.3%	5.4%	4.6%	2.9%	0.02
$E[|\Delta p|]$ for US Data:	-	8.5%	7.75%	6.75%	6.5%	-	-

Value of $\frac{\psi_1}{B\sigma^2}$ chosen to match the size of price changes at $n = 2$. US data from Bhattarai and Schoenle (2010), Figure 4. Implied ψ_1 using $B = 20$ and $\sigma = 0.15$.

and of their absolute value $|\Delta p|$ depend only on n *and* $\bar{y}$. Thus, any normalized statistics such as ratio of moments (kurtosis, skewness, etc) or a ratio of points in the c.d.f. depends exclusively on n. Indeed the kurtosis is given in Proposition 7, as $Kurtosis(\Delta p_i) = 3n/(2+n)$, which is an increasing concave function, starting at 1 and converging to 3. Table 4 uses the expressions of the model to compute several moments of interest. These moments have been estimated using two scanner data sets by Midrigan (2009) and also using BLS producer data by Bhattarai and Schoenle (2010). A summary of the selected statistics from these papers is reproduced in Table 5.

We briefly comment on the reasons why the statistics chosen in Table 4 with Table 5 are of interest. Note that the case of $n = 1$, price changes are binomial, either $-\sqrt{\bar{y}}$ or $+\sqrt{\bar{y}}$ with the same probability, so its absolute value has a degenerate distribution. As the the number of goods increases the dispersion of the absolute value increases. The distribution includes larger price changes, so that its kurtosis also increases with n. As there are more goods, some goods will be adjusted even if their price is almost optimal, and hence the fraction of small price changes increases with n. We draw two conclusions from the comparison of Table 4 with Table 5. First, for the four moments computed our model falls short from the data. In particular, as shown in Proposition 7 the distribution in the model converged to a normal as n goes to ∞. Yet the data displays values for the four moments even larger than the ones

Table 4: Statistics for price changes as function of number of products, Model economy

statistics \ number of products n	1	2	3	4	5	6	10	20	50
$Std(\|\Delta p_i\|)\ /\ E(\|\Delta p_i\|)$	0	0.48	0.58	0.62	0.65	0.67	0.70	0.74	0.75
Kurtosis(Δp_i)	1.0	1.7	2.0	1.9	2.1	2.3	2.5	2.8	2.9
Fraction: $\|\Delta p_i\| < \frac{1}{2}E(\|\Delta p_i\|)$	0	0.21	0.25	0.27	0.28	0.28	0.30	0.31	0.31
Fraction: $\|\Delta p_i\| < \frac{1}{4}E(\|\Delta p_i\|)$	0	0.10	0.12	0.13	0.14	0.14	0.15	0.16	0.16

Δp_i denotes the log of the price change, and $|\Delta p_i|$ the absolute value of the log of price changes. They are computed using the results in Proposition 7. All statistics in the table depend exclusively on n. Kurtosis defined as the fourth moment relative to the square of the second.

corresponding to a standard normal. Second, our model reproduces the pattern of the four moments in terms of their variation with respect to the number of products n.

Table 5: Statistics for price changes as function of the number of products, US data

Statistics	*Bhattarai and Schoenle*				*Midrigan*			
	Number of products n				AC Nielsen		Dominick's	
	2	4	6	10	All	No Sales	All	No Sales
$Std(\|\Delta p_i\|)\ /\ E(\|\Delta p_i\|)$	1.02	1.15	1.30	1.55	0.68	0.72	0.84	0.81
Kurtosis(Δp_i)	5.5	7.0	11	17	3.0	3.6	4.1	4.5
Fraction: $\|\Delta p_i\| < \frac{1}{2}E(\|\Delta p_i\|)$	0.39	0.45	0.47	0.50	0.24	0.25	0.34	0.31
Fraction: $\|\Delta p_i\| < \frac{1}{4}E(\|\Delta p_i\|)$	0.27	0.32	0.35	0.38	0.10	0.10	0.17	0.14

Sources: For the Bhattarai and Schoenle (2010) data: the number of product n is the mean of the categories considered based on the information in Table 1, the ratio $Std(|\Delta p_i|)\ /\ E(|\Delta p_i|)$ is from Table 2 (Firm-Based), the fraction of $|\Delta p_i|$ which are small is from Table 14, the Kurtosis is from Figure 7. The data from Midrigan (2007) are taken from distribution of standardized prices in Table 2a.

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A Proofs

PROPOSITION 8. The origin is the optimal return point.

Proof. By contradiction, suppose it is not the case, and assume w.l.o.g. that $t = 0$ is a period where and adjustment takes place and that $\hat{p}_i > 0$. Then, consider an alternative plan where $\hat{p}_i' = 0$ and where $\mathcal{I}' = \mathcal{I} + \{(0, 0, ..., -\hat{p}_i, ..., 0)\}$ so that the next adjustment happens exactly with the same probabilities. Since between $0 \le t \le \tau$ we have $\mathbb{E}_0[||p'(t)||^2] = \mathbb{E}_0[||p(t)||] + \hat{p}_i^2$, and thus setting $\hat{p}_i > 0$ is not optimal.

Proof. (of Proposition 2) Notice that $v'(0) = \beta_1$ and that $v(0) = \beta_0$, so that we require $\beta_1 > 0$, which implies $\beta_0 > 0$. Moreover, if $\beta_1 > B/r$ then v is strictly increasing and strictly convex. If $\beta_1 = B/r$ then v is linear in y. If $0 < \beta_1 < B/r$, then v is strictly increasing at the origin, strictly concave, and it reaches its unique maximum at a finite value of y. Thus, a solution that satisfies smooth pasting requires that $0 < \beta_1 < B/r$, and the maximizer is $\bar{y}$. In this case, $y = 0$ achieves the minimum in the range $[0, \bar{y}]$. Thus we have verified i), ii) and iii). Finally, we require value matching at $\bar{y}$, i.e. $v(\bar{y}) = v(0) + \psi$. Let $\beta_i(\beta_1)$ be the solution of equation (15), as a function of β_1. Note that for $0 < \beta_1 < B/r$, all the $\beta_i(\beta_1) < 0$ for $i \geq 2$ and are increasing in β_1, converging to zero as β_1 goes to B/r. Smooth pasting can be written as

$$0 \;\; = \;\; v'(\bar{y}; \beta_1) \equiv \sum_{i=1}^{\infty} \; i \; \beta_i(\beta_1) \; \bar{y}^{i-1} \; ,$$

where we emphasize that all the β_i can be written as a function of β_1. From the properties of the $\beta_i(\cdot)$ discussed above, it follows that we can write the unique solution of $0 = v'(\bar{\rho}(\beta_1); \beta_1)$ as an strictly increasing function of β_1, i.e. $\bar{\rho}'(\beta_1) > 0$. Now we write value matching at $\bar{y}$ which gives:

$$\psi \;\; = \;\; v(\bar{y}, \beta_1) - v(0, \beta_1) = v(\bar{y}, \beta_1) - \beta_0(\beta_1) = \sum_{i=1}^{\infty} \; \beta_i(\beta_1) \; \bar{y}^i \; .$$

We note that, given the properties of $\beta_i(\cdot)$ discussed above, for *any given* $y > 0$ we have: $v(y, \beta_1) - \beta_0(\beta_1)$ is strictly increasing in β_1, as long as $0 < \beta_1 < B/r$. Thus, define

$$\Psi(\beta_1) \;\; = \;\; v(\bar{\rho}(\beta_1), \beta_1) - v(0, \beta_1) = \sum_{i=1}^{\infty} \; \beta_i(\beta_1) \; \bar{\rho}(\beta_1)^i \; .$$

From the properties discussed above we have that $\Psi(\beta_1)$ is strictly increasing in β_1 and that it ranges from 0 to ∞ as β_1 ranges from 0 to B/r. Thus Ψ is invertible. The solution of the problem is given by setting:

$$\beta_1(\psi) = \Psi^{-1}(\psi) \qquad \text{and} \qquad \bar{y}(\psi) = \bar{\rho}(\beta_1(\psi)) \; .$$

Now we show that the solution of the problem, as $r \downarrow 0$ satisfies:

$$\bar{y} = \sigma^2 \; Q\left(\frac{\psi}{B \; \sigma^2}, n\right) \; .$$

for some increasing function $Q(\cdot)$. As $r \downarrow 0$ solving the expected discounted problem is equivalent to solve the following steady state problem. The objective is to chose a threshold value of $\bar{y}$ in order to minimize the sum of two terms. The first is expected number of adjustment times the cost per adjustment ψ from following the threshold policy. We denote this term by $N_a(\bar{y}; \sigma^2 n)\psi$. The second is the expected value of the deviations, times the cost of each deviation B. We denote this term by $e(\bar{y};\sigma^2,n)B$. The minimization problem is then

$$\min_{\bar{y}} N_a(\bar{y};\sigma^2 n)\ \psi + e(\bar{y};\sigma^2,n)\ B$$

To be clear about the definition of this problem, denote $F(\cdot,n)$ the measure on the sample paths of n independent standard BM's, starting at zero at time zero. Let $\tau(\omega;\gamma,n)$ the first time that the sum of the squares of the BM's hits γ, and let $A(\gamma,n)=\{\omega : \sum_{i=1}^n W_i^2(\tau,\omega)=\gamma\}$ be the sample paths for which the sum of the squares hit γ. We have that

$$N_a(\bar{y};\sigma^2 n) = \frac{1}{\int_{\omega\in A\left(\frac{\bar{y}}{\sigma^2},n\right)} \tau(\omega;\frac{\bar{y}}{\sigma^2},n)F(\omega,n)d\omega} = \frac{n\ \sigma^2}{\bar{y}}$$

where the last computation follow from equation (18) obtained in Proposition 4.

$$e(\bar{y};\sigma^2,n) \equiv \sigma^2 \int_{\omega\in A\left(\frac{\bar{y}}{\sigma^2},n\right)} \frac{\int_0^{\tau\left(\omega;\frac{\bar{y}}{\sigma^2},n\right)} \sum_{i=1}^n W_i^2(t,\omega)dt}{\tau\left(\omega;\frac{\bar{y}}{\sigma^2},n\right)} F(\omega;n)d\omega$$

We note that since the n BM's have a normal finite distribution and are assumed to be independent, we have that

$$e(\bar{y};\sigma^2,n) = \sigma^2 e\left(\frac{\bar{y}}{\sigma^2};1,n\right).$$

We can write the minimization problem as:

$$B\min_{\bar{y}} \frac{n\ \sigma^2}{\bar{y}}\frac{\psi}{B} + \sigma^2 e\left(\frac{\bar{y}}{n\sigma^2};1,n\right) = \sigma^2 B \min_{\tilde{y}} \frac{n}{\tilde{y}}\frac{\psi}{B\ \sigma^2} + e\left(\tilde{y};1,n\right)$$

where $\tilde{y}\equiv\bar{y}/\sigma^2$. Let the solution of the transformed problem be $\tilde{y}=Q\left(\frac{\psi}{B\ \sigma^2},n\right)$ then the solution of the original problem is:

$$\bar{y} = \sigma^2\ Q\left(\frac{\psi}{B\ \sigma^2},n\right)\ .$$

That the function $Q(\cdot)$ is strictly increasing follows from the previous result and since ψ is an argument of this function.

Theorem 1. Øksendal (2000) Theorem 10.4.1 adds the following to equations (3)-(6) to show that a function verifying these conditions is the solution of the problem.

1. $0 \le V(p) \le A(p)$ for all $p\in\mathbb{R}^n$ where $A(p) = B||p||^2\ n\ (\sigma/r)^2$ is the expected discounted value of never-adjusting,
2. $rV(p) \le B\ ||p||^2 + \frac{\sigma^2}{2}\sum_{i=1}^n V_{ii}(p)$ for all $p\in\mathbb{R}^n\backslash\bar{\mathcal{I}}$,
3. $V(p) \le \min_{\hat{p}} V(\hat{p}) + \psi$ for all $p\in\mathbb{R}^n$,
4. $\partial\mathcal{I}$ is a Lipschitz surface: i.e. it is locally the graph of an Lipschitz function,

5. the process $\{p(t)\}$ spends no time in the boundary of the inaction region:

$$\mathbb{E}\left[\int_0^\infty \chi_{\{\partial\mathcal{I}\}}(p(t))\ dt\ |p(0)=p\right]=0,\ \text{for all } p\in\mathbb{R}^n\ .$$

6. The second derivatives of V are bounded in a neighborhood of $\partial\mathcal{I}$,

7. the stopping times τ_i^* that achieve the solution are finite,

8. Let τ^* be the optimal stopping time starting from $p(0)$, the family $\{e^{-r\tau}V(p(\tau));\tau\le\tau^*\}$ is uniformly integrable for all $p(0)$.

For completeness we state the definition of a Lipschitz surface.

Definition 1. The boundary of a bounded set $\mathcal{I}\subset\mathbb{R}^n$ denoted by $\partial\mathcal{I}$ has Lipschitz domain (or it is a Lipschitz surface) if there is constant $K>0$ such that for all $p\in\partial I$ there is a neighborhood $B_\epsilon(p)\cap\mathcal{I}$ and a system of coordinates $x=(p_1,p_2,...,p_{n-1}),y=p_n$ and a function h_p such that for all:

1. $|h(x_1)-h(x_2)|<K|x_1-x_2|$ for all x_1,x_2,
2. $B_\epsilon(p)\cap\mathcal{I}=B_\epsilon(p)\cap\{(x,y):y>h_p(x)\}$, and
3. $B_\epsilon(p)\cap\partial\mathcal{I}=B_\epsilon(p)\cap\{(x,y):y=h_p(x)\}$.

Proof. (of Proposition 3) We show that V so constructed has the following properties:

1. it only depends on the absolute value of the prices, since for all $p\in\mathbb{R}^n$:

$$v\left(\sum_{i=1}^n p_i^2\right)=v\left(\sum_{i=1}^n |p_i|^2\right)\ .$$

   for all $p\in\mathbb{R}^n$,

2. The range of inaction is given by $\mathcal{I}=\{p\in\mathbb{R}^n\big|\sum_{i=1}^n p_i^2\le\bar{y}\}$.

3. It solves the ODE given by equation (3). This can be seen by computing:

$$V_i(p)=v'\left(\sum_{i=1}^n p_i^2\right)2p_i\ \text{ and }\ V_{ii}(p)=v''\left(\sum_{i=1}^n p_i^2\right)(2p_i)^2+v'\left(\sum_{i=1}^n p_i^2\right)2\ ,$$

   replacing this into the ODE equation (3) we obtain the ODE equation (10), which v solves by hypothesis.

4. It satisfies value matching equation (4), which is immediate since it satisfied the value matching condition for v given in equation (11).

5. it satisfies smooth pasting equation (6). Using the form of the solution for v, namely:

$$V_i(p) = v'\left(\sum_{i=1}^{n} p_i^2\right) 2\ p_i = \sum_{j=1}^{n}\ \beta_j\ j\ \left(\sum_{k=1}^{n} p_k^2\right)^{j-1}\ 2\ p_i$$

Using that v satisfies smooth pasting we have:

$$0 = \sum_{j=1}^{n}\ \beta_j\ j\ \left(\sum_{k=1}^{n} p_k^2\right)^{j-1}$$

for any p with $\sum_{k=1}^{n} p_k^2 = \bar{y}$, which establishes that $V_i(p) = 0$ for all $i = 1, .., n$ and for any $p \in \partial\mathcal{I}$.

6. It satisfies optimality of the origin as return point, as given by equation (5). Direct computation gives:

$$V_i(p) = v'\left(\sum_{i=1}^{n} p_i^2\right) 2\ p_i = \sum_{j=1}^{n}\ \beta_j\ j\ \left(\sum_{k=1}^{n} p_k^2\right)^{j-1}\ 2\ p_i$$

which equals zero when evaluated at $p = 0$. Notice also that

$$V_{ii}(0) = 2\ \beta_1 > 0\ \text{ for all } i = 1, ..., n\ \text{ and }\ V_{ij}(0) = 0\ ,$$

thus, $p = 0$ is a local minimum.

Finally we show that a function V with these properties is a strong solution to the variational inequality of the problem, and hence it is the value function by checking the extra conditions of Theorem 1.
Item (1) holds by construction of v as in Proposition 2, were we have for all $y > 0$ or $p \neq 0$:

$$V(p) = v(y) > v(0) = \beta_0 = \frac{n\sigma^2}{r}\beta_1 < \frac{Bn\sigma^2}{r^2} = A(p)\ .$$

Item (2) holds with equality in $\mathcal{I}$ by construction. It holds as inequality in $Int(\mathcal{C})$. To see why

$$V_i(p) = v'(||p||^2)2p_i\ \text{ and }\ V_{ii}(p) = v''(||p||^2)4(p_i)^2 + v'(||p||^2)2\ ,$$

but using Proposition 2 at $p \in \partial\mathcal{I}$ we have $v'(||p||^2) = 0$ and $v''(||p||^2) < 0$. Additionally, $||p||^2 > \bar{y} \in Int(\mathcal{C})$, thus

$$rV(\bar{y}) = B\bar{y} + \frac{\sigma^2}{2}v''(\bar{y})n4\bar{y} < rV(p) = B||p||^2 \text{ for all } p \in Int(\mathcal{C})\ .$$

Item (3) holds since by Proposition 2 we have $v(y)$ is strictly increasing in $(0, \bar{y})$ and $v(\bar{y}) - v(0) = \psi$, thus $V(p) > V(0) + \psi$ for all $p \neq 0$.

Item (4) holds by taking $h(p_1, p_2, ..., p_{n-1}) = \sqrt{\bar{y} - \sum_{i=1}^{n-1} p_i^2}$ for $p_n^2 > 0$, otherwise take a different coordinate system, i.e. solve for the ith coordinate for which $p_i^2 > 0$. Clearly h is Lipschitz.
Item (5) holds by considering the uncontrolled process $\mathrm{d}y = n\sigma^2 \mathrm{d}t + 2\sigma\sqrt{y}\ \mathrm{d}W$ and thus $\mathbb{E}_0[y(t)] = n\sigma^2 t + y(0)$.
Item (6) holds since, as shown above, $V_{ii}(p) = v''(\bar{y})\ 4\ p_i^2$ and

$$v''(\bar{y}) = \sum_{i=2}^{\infty} \beta_i\ i\ (i-1)\ (\bar{y})^{i-2}$$

and since, as shown in Proposition 1, $\lim_{i\to\infty} \beta_{i+1}/\beta_i = 0$ and thus the function v is analytical for all $y > 0$.
Item (7) holds, since $y(t)$ has a strictly positive drift $n\sigma^2$.
Item (8) holds since $e^{-r\tau}V(p(\tau)) \leq e^{-r\tau^*}(\psi + V(0))$.

Proof. (of Proposition 5) The forward Kolmogorov equation is:

$$0 = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left([2\sigma\sqrt{y}]^2 f(y)\right) - \frac{\partial}{\partial y}\left(n\sigma^2 f(y)\right) \quad \text{for } y \in (0, \bar{y})\ , \tag{26}$$

with boundary conditions:

$$1 = \int_0^{\bar{y}} f(y)\ dy \quad \text{and} \quad f(\bar{y}) = 0\ . \tag{27}$$

The first boundary conditions ensures that f is a density. The second is implied by the fact that when the process reaches $\bar{y}$ it is return to the origin, so the mass escape from these points. Equation (27) implies the second order ODE: $f'(y)(\frac{n}{2} - 2) = yf''(y)$. The solution of this ODE for $n \neq 2$ is $f(y) = A_1 y^{n/2-1} + A_0$ for two constants A_0, A_1 to be determine using the boundary conditions equation (27):

$$\begin{aligned} 0 &= A_1(\bar{y})^{n/2-1} + A_0\ , \\ 1 &= \frac{A_1}{n/2}(\bar{y})^{n/2} + A_0\bar{y}\ . \end{aligned}$$

For $n = 2$ the solution is $f(y) = -A_1 \log(y) + A_0$ subject to the analogous conditions. Solving for the coefficients A_0, A_1 gives the desired expressions.

Proof. (of Proposition 6) Let τ be the stopping time defined by the first time where the sum of the square of the price gaps vector $||p(\tau)||^2$ reaches the critical value $\bar{y}$, starting at the origin at time zero, i.e. starting at $||p(0)|| = 0$. Let $S_n(t, \bar{y})$ be the probability distribution for stopping times $\tau \geq t$, alternatively let $S_n(\cdot, \bar{y})$ be the survival function. Theorem 2 Ciesielski and Taylor (1962) shows that for $n \geq 1$:

$$S_n(t, \bar{y}) = \sum_{k=1}^{\infty} \xi_{n,k} \exp\left(-\frac{q_{n,k}^2}{2\bar{y}}\ \sigma^2\ t\right)\ , \text{ where } \xi_{n,k} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{q_{n,k}^{\nu-1}}{J_{\nu+1}(q_{n,k})}\ . \tag{28}$$

where $J_\nu(z)$ is the Bessel function of the first kind, where $\nu = (n-2)/2$, where $q_{n,k}$ are the positive zeros of the Bessel function $J_\nu(z)$, index in ascending order according to k, and where Γ is the gamma function. The hazard rate is then given by:

$$h_n(t,\bar{y}) = -\frac{1}{S_n(t,\bar{y})}\frac{\partial S_n(t,\bar{y})}{\partial t}, \text{ with asymptote } \lim_{t\to\infty} h_n(t,\bar{y}) = \frac{q_{n,1}^2\,\sigma^2}{2\,\bar{y}}\ . \tag{29}$$

As shown by Qu and Wong (1999), the zeroes of the Bessel function $q_{n,k}$ satisfy for $n > 2$ the following inequalities:

$$\left(\frac{n}{2}-1\right) - \frac{a_k}{2^{1/3}}\left(\frac{n}{2}-1\right)^{1/3} < q_{n,k} < \left(\frac{n}{2}-1\right) - \frac{a_k}{2^{1/3}}\left(\frac{n}{2}-1\right)^{1/3} + \frac{3}{10}a_k^2\frac{2^{1/3}}{\left(\frac{n}{2}-1\right)^{1/3}} \tag{30}$$

where a_k are the first negative zero of the Airy function. For instance $a_1 \approx -2.33811$, giving a tight bound for the first zero $q_{n,1}$, which determines the asymptote of the hazard rate. A related simpler lower bound given by Hethcote (1970) for $n \geq 2$ is

$$q_{n,k}^2 > \left(k - \frac{1}{4}\right)^2 \pi^2 + \left(\frac{n}{2}-1\right)^2\ . \tag{31}$$

Proof. (of Proposition 7) We first establish the following Lemma. Lemma. Let z be distributed uniformly on the surface in the surface of the n-dimensional sphere of radius one. We use x for the projection of z in any of the dimension, so $z_i = x \in [-1,1]$. The marginal distribution of $x = z_i$ has density:

$$\begin{aligned} f_n(x) &= \int_0^\infty \frac{s^{(n-3)/2}e^{-s/2}}{2^{(n-1)/2}\,\Gamma[(n-1)/2]}\frac{e^{-sx^2/[2(1-x^2)]}}{\sqrt{2\pi}}\frac{s^{1/2}}{(1-x^2)^{3/2}}\,ds \\ &= \frac{\Gamma(n/2)}{\Gamma(1/2)\,\Gamma[(n-1)/2]}(1-x^2)^{(n-3)/2} \end{aligned} \tag{32}$$

where the Γ function makes the density integrate to one. This lemma is an application of Theorem 2.1, part 1 in Song and Gupta (1997), setting $p = 2$, so it is euclidian norm, and $k = 1$ so it is the marginal of one dimension. We give a simpler proof below.

Now we consider the case where the sphere has radius different from one. Let $p \in \partial\mathcal{I}$, then

$$p = \frac{p}{\sum_{i=1}^n p_i^2}\bar{y} = \frac{p}{\sqrt{\sum_{i=1}^n p_i^2}}\ \sqrt{\bar{y}} = z\ \sqrt{\bar{y}}$$

where z is uniformly distributed in the n dimensional sphere of radius one. Thus each p_i has the same distribution than $x\ \sqrt{\bar{y}}$. Using the change of variable formula we obtained the required result.

Part 2 of Theorem 2.1 in in Song and Gupta (1997) shows that if x the marginal of a uniform distributed vector in the surface of the n-dimensional sphere, then x^2 is distributed as a $Beta(\frac{1}{2}, n-12)$. If y is distributed as a $Beta(\alpha,\beta)$ then it has $\mathbb{E}(y) = \alpha/(\alpha+\beta)$ and $\mathbb{E}(y^2) = (\alpha+1)/(\alpha+\beta+1)\mathbb{E}(y)$. Using these expressions for $\alpha = 1/2$ and $\beta = n/2$ we obtain the results for the standard deviation of Δp_i and its kurtosis. For the expected value of the absolute value of price changes we note that

$$
\begin{aligned}
\mathbb{E}[|\Delta p_i|] &= 2\int_0^{\sqrt{y}} \Delta p_i \; w(\Delta p_i) \; d\Delta p_i \\
&= \frac{2}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)} \int_0^{\sqrt{y}} \Delta p_i \left(1-\left(\frac{\Delta p_i}{\sqrt{y}}\right)^2\right)^{(n-3)/2} \qquad d\Delta p_i \\
&= \frac{\sqrt{y}}{\frac{n-1}{2}\; Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}
\end{aligned}
$$

where the second line uses the form of h and the last line uses that the following result:

$$
\int_a^b x\left(1-x^2\right)^{(n-3)/2} dx = \left.\frac{\left(1-x^2\right)^{(n-1)/2}}{1-n}\right|_a^b .
$$

Then we have, using the fundamental property of the Gamma function

$$
\frac{1}{\frac{n-1}{2}\; Beta\left(\frac{n-1}{2},\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{n}{2}\right)}{\frac{n-1}{2}\;\Gamma\left(\frac{n-1}{2}\right)\;\Gamma(1/2)} = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\;\Gamma(1/2)}
$$

Thus

$$
\mathbb{E}[|\Delta p_i|] = \frac{\sqrt{y}}{\frac{n-1}{2}\; Beta\left(\frac{n-1}{2},\frac{1}{2}\right)} = \sqrt{y}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\;\Gamma(1/2)}
$$

We can approximate these ratio of Gamma functions as

$$
\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\;\Gamma(1/2)} \approx \sqrt{\frac{2}{\pi}}\frac{\sqrt{n+1/2}}{n}
$$

from where we obtain our expression.

For $Std\,(\;|\Delta p_i|\;)\;/\;\mathbb{E}\,(\;|\Delta p_i|\;)$ we use that, given the symmetry around zero we have:

$$
\begin{aligned}
Std\,(\;|\Delta p_i|\;)\;/\;\mathbb{E}\,(\;|\Delta p_i|\;) &= \sqrt{\frac{\mathbb{E}[\Delta p_i^2]}{\mathbb{E}[|\Delta p_i|]^2}-1} = \sqrt{\left(\frac{Std(\Delta p_i)}{\mathbb{E}[|\Delta p_i|]}\right)^2-1} \\
&= \sqrt{\left(\frac{\frac{n-1}{2}\; Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}{\sqrt{n}}\right)^2-1} \approx \sqrt{\frac{\pi}{2}\left(\frac{2n}{1+2n}\right)-1}
\end{aligned}
$$

For the convergence of $\Delta p_i / Std(\Delta p_i)$ to a normal, we show that $y = x^2\; n$ converges to a chi-square distribution with 1 d.o.f., where x is the marginal of a uniform distribution in the surface of the n-dimensional sphere. The p.d.f of $y \in [0, n]$, the square of the standardized x, is

$$
\frac{\Gamma\left(\frac{n}{2}\right)}{n\;\Gamma\left(\frac{n-1}{2}\right)\;\Gamma\left(\frac{1}{2}\right)}\left(1-\left(\frac{y}{n}\right)\right)^{(n-3)/2}\left(\frac{y}{n}\right)^{-1/2},
$$

and the p.d.f. of a chi-square with 1 d.o.f. is

$$\frac{\exp(-y/2)\ y^{-1/2}}{\sqrt{2}\,\Gamma\left(\frac{1}{2}\right)}\ .$$

Then, fixing y, taking logs in the ratio of the two p.d.f.'s, and taking the limit as $n \to \infty$, using that

$$\frac{\Gamma\left(\frac{n}{2}\right)\ \sqrt{2}}{\Gamma\left(\frac{n-1}{2}\right)\ \sqrt{n}} \to 1 \text{ as } n \to \infty\ .$$

we obtain that the ratio of the two p.d.f.'s converges to one.

B Derivation of the approximation for $\bar{y}$

The smooth pasting and value matching conditions gives a system of two equations in two unknowns,

$$\begin{aligned} 0 &= v'(\bar{y}) = \beta_1 + 2\beta_2\bar{y} \\ \psi &= v(\bar{y}) - v(0) = \beta_1\bar{y} + \beta_2\bar{y}^2\ , \end{aligned}$$

where $\beta_2 = (r\beta_1 - B)/(2\sigma^2(n+2))$. We replace $\bar{y}$ and β_2 to obtain the following quadratic solution for β_1:

$$-\beta_1^2 - \rho\beta_1\left(\frac{2\ \psi}{\sigma^2\ (n+2)}\right) + \frac{B\ \psi\ 2}{\sigma^2\ (n+2)} = 0$$

taking the positive root and replacing it back to solve for $\bar{y}$ yields:

$$\bar{y} = \frac{2\psi}{-\frac{r\psi}{\sigma^2(n+2)} + \sqrt{\left(\frac{r\psi}{\sigma^2(n+2)}\right)^2 + \frac{2B\psi}{\sigma^2(n+2)}}}$$

Taking r to zero in this expression we obtain equation (16).

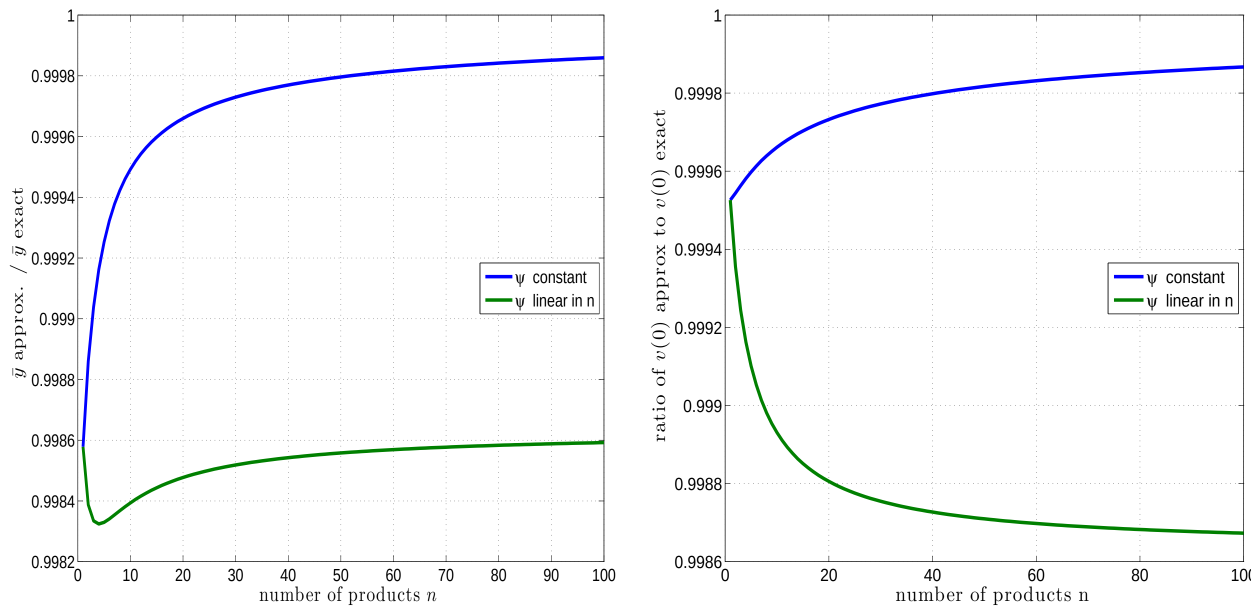
C Numerical accuracy of the approximation

In this section we present some evidence on the numerical accuracy of the approximation. We compare the value of $\bar{y}$ obtained from the quadratic approximation to v described above, with what we call the "exact" solution, which is the numerical solution using up to 30 terms for β_i in its the expansion.

The approximation are closer for smaller values of σ and ψ, which we regard as more realistic.

The next figure shows the value of $N_a(n)$ for various n when the menu cost are constant returns to scale, so $\psi = \psi_1 n$ using the approximation and using the "exact" expression.

Figure 4: Ratio of $\bar{y}'s$ and of $v(0)'s$ for the approximation relative to the "exact" solution

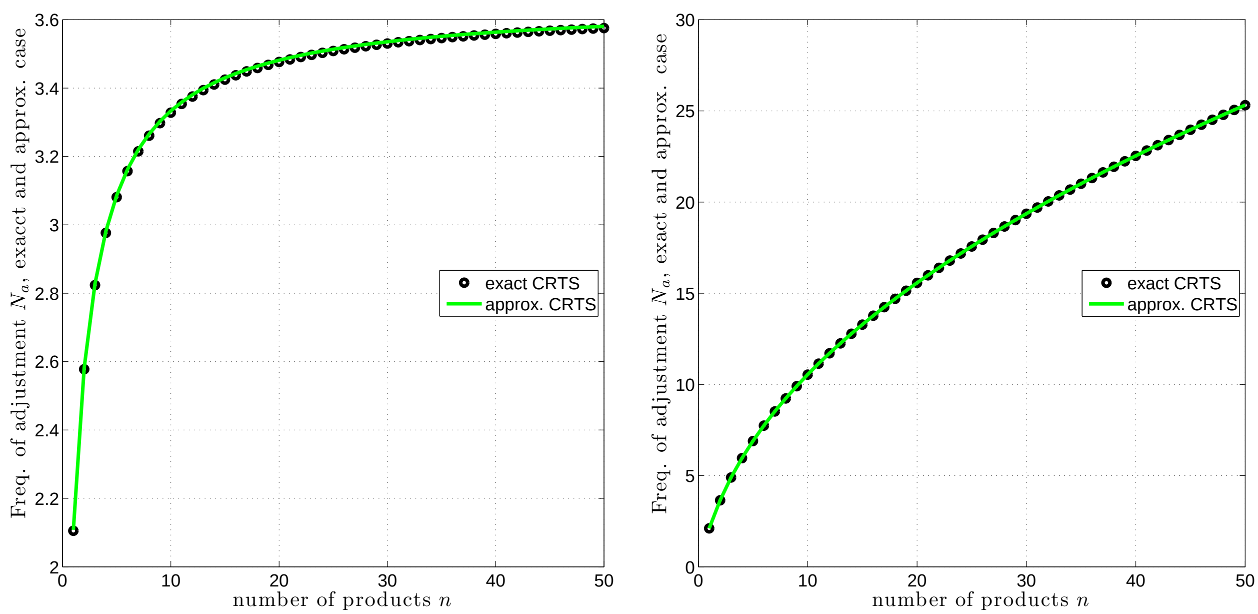


Note: parameter values are $B = 20$, $\sigma = 0.25$, $\psi_1 = 0.03$ and $r = 0.03$.

Figure 5: Frequency of adjustment N_a for the CRTS $\psi_1 n$ and constant ψ.

CRTS: ψ proportional to n

Constant ψ



Note: parameter values are $B = 20$, $\sigma = 0.20$, $\psi_1 = 0.03$ and $r = 0.03$.

D Approximating the Profit Function

Consider the expression for the rate of profits of a multiproduct firm. The marginal cost for each of the products is C_i. The demand system is given by the sum of n independent demands, with own price elasticity given by η. The parameter A_i is the intercept, in logs, of the demand for the $i-th$ product. Given the constant elasticity of demand and constant marginal cost, the frictionless optimal price for the monopolist is a multiple of the marginal cost, and independent of A_i. To keep the n goods symmetric we will assume that C_i and A_i are perfectly correlated, so that when cost are high, and hence frictionless prices are high, demand is also high. In this way we can keep the share of profits coming to each of the n goods comparable, even if cost differ significantly.

We write the total profits per product

$$\Pi(P_1,..,P_n,C_1,...,C_n,A_1,...,A_n) \equiv \sum_{i=1}^{n} \Pi(P_i,C_i,A_i) = \sum_{i=1}^{n} A_i \; P_i^{-\eta} \left(P_i - C_i\right)$$

Let $P_i^* = \arg\max_P \Pi(P,C_i,A_i)$. Assuming that

$$A_i = A \; (C_i)^{\eta-1} \; ,$$

we obtain that profits, relative to the maximized profits, can be written as

$$\begin{aligned}
&\frac{\Pi(P_1,..,P_n,C_1,...,C_n,A_1,...,A_n) - \Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)}{\Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} \\
&= B \sum_{i=1}^{n} \left(\frac{P_i - P_i^*}{P_i^*}\right)^2 + o\left(\sum_{i=1}^{n}\left(\frac{P_i - P_i^*}{P_i^*}\right)^2\right)
\end{aligned}$$

where $B = \frac{(\eta-1)\eta}{2\;n}$.

To obtain the quadratic expression above we write a second order expansion of the profits, divide both sides by the maximized total profits, and complete elasticities:

$$\begin{aligned}
&\frac{\Pi(P_1,..,P_n,C_1,...,C_n,A_1,...,A_n)}{\Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} \\
&= 1 + \sum_{i=1}^{n} \frac{1}{\Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} \frac{\partial}{\partial P_i} \left.\Pi\left(P_i,C_i,A_i\right)\right|_{P_i^*} P_i^* \left(\frac{P_i - P_i^*}{P_i^*}\right) \\
&+ \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} \frac{\partial^2}{\partial P_i^2} \left.\Pi\left(P_i,C_i\right)\right|_{P_i^*} (P_i^*)^2 \left(\frac{P_i - P_i^*}{P_i^*}\right)^2
\end{aligned}$$

Computing the derivatives for our functional forms:

$$
\begin{aligned}
\frac{\partial}{\partial P_i}\Pi\left(P_i,C_i,A_i\right) &= A_i P^{-\eta}\left(-\eta\left(\frac{P_i-C_i}{P_i}\right)+1\right)\\
\frac{\partial^2}{\partial P_i^2}\Pi\left(P_i,C_i\right) &= -A_i P^{-\eta}\eta\frac{1}{P_i}\left(-\eta\left(\frac{P_i-C_i}{P_i}\right)+1\right)-A_i P^{-\eta}\eta\left(\frac{C_i}{P_i^2}\right)
\end{aligned}
$$

We have the standard result of a constant mark-up:

$$
P^* = \frac{\eta}{\eta-1}C_i \implies \left(-\eta\left(\frac{P_i-C_i}{P_i}\right)+1\right) = \left(-\eta\left(\frac{\eta C_i - C_i(\eta-1)}{\eta C_i}\right)+1\right) = 0\ ,
$$

and the maximized value of profits given by

$$
\Pi(P_i^*,C_i,A_i) = A_i C_i^{-\eta}\left(\frac{\eta}{\eta-1}\right)^{-\eta} C_i\left(\frac{1}{\eta-1}\right) = A_i\ C_i^{1-\eta}\left(\frac{\eta}{\eta-1}\right)^{-\eta}\left(\frac{1}{\eta-1}\right)\ .
$$

Hence the first and second derivatives, evaluated a the optimal prices are:

$$
\begin{aligned}
\frac{\partial}{\partial P_i}\Pi\left(P_i,C_i,A_i\right)|_{P^*} &= 0\\
\frac{\partial^2}{\partial P_i^2}\Pi\left(P_i,C_i,A_i\right)|_{P^*} &= -A_i P^{*-\eta}\eta\ \frac{C_i}{P_i^{*2}} = -A_i\left(C_i\frac{\eta}{\eta-1}\right)^{-\eta}\ \frac{\eta C_i}{P_i^{*2}}
\end{aligned}
$$

and

$$
\frac{1}{\Pi\left(P_i^*,C_i,A_i\right)}\frac{\partial^2}{\partial P_i^2}\Pi\left(P_i,C_i,A_i\right)|_{P^*}\left(P_i^*\right)^2 = -\frac{A_i\left(C_i\frac{\eta}{\eta-1}\right)^{-\eta}\ \eta C_i}{A_i\ C_i^{1-\eta}\left(\frac{\eta}{\eta-1}\right)^{-\eta}\left(\frac{1}{\eta-1}\right)} = -\left(\eta-1\right)\eta
$$

Thus the expansion can be written as:

$$
\begin{aligned}
&\frac{\Pi(P_1,..,P_n,C_1,...,C_n,A_1,...,A_n)}{\Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)}\\
&= 1+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)}\frac{\partial^2}{\partial P_i^2}\ \Pi\left(P_i,C_i\right)|_{P_i^*}\left(P_i^*\right)^2\left(\frac{P_i-P_i^*}{P_i^*}\right)^2\\
&= 1-\frac{1}{2}\sum_{i=1}^{n}\frac{\Pi(P_i^*,C_i,A_i)}{\Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)}\left(\eta-1\right)\eta\left(\frac{P_i-P_i^*}{P_i^*}\right)^2
\end{aligned}
$$

Using the assumption that $A_i = A\ (C_i)^{\eta-1}$ we have that

$$
\frac{\Pi(P_i^*,C_i,A_i)}{\Pi(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} = \frac{A_i C_i^{1-\eta}}{\sum_{j=1}^{n}A_j C_j^{1-\eta}} = \frac{1}{n}\ ,
$$

and hence the expression for B is:

$$B = \frac{(\eta - 1)\eta}{2 \quad n} \ .$$