Existence and Indeterminacy of Markovian Equilibria in Dynamic Bargaining Games

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Abstract

We show that many dynamic bargaining games of interest are characterized by a continuum of stationary Markov perfect equilibria. In fact, when the set of alternatives is high-dimensional and players are patient, there is a continuum of equilibria close to any alternative satisfying a simple linear independence condition on the players' gradients. This condition applies to a wide range of economic environments, such as pie-division settings and, more generally, all economies with a private good component. The approach extends the simple solutions approach of Anesi and Seidmann (2015) to the spatial setting. The implication is that constructive techniques, which involve an explicit specification of a particular equilibrium and are common in the literature, implicitly rely on a restrictive selection of equilibria.

1 Introduction

Most formal political analysts of legislative policymaking, until recently, have used models in which legislative interaction ends once a proposal is passed (e.g., Romer and Rosenthal 1978, Baron and Ferejohn 1989, and Banks and Duggan 2000, 2006). As pointed out by Baron (1996) and later by Kalandrakis (2004), however, most legislatures have the authority to change existing laws by enacting new legislation; so that laws continue in effect only in the absence of new legislation. To explore this dynamic feature of legislative policymaking, these authors have introduced an alternative model that casts the classical spatial collective-choice problem into a dynamic bargaining framework. Each period begins with a status quo policy inherited from the previous period, and a legislator is chosen randomly to propose any feasible policy, which is then subject to an up or down vote. If the proposal is voted up, then it is implemented in that period and becomes the next

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period's status quo; if it is voted down, then the ongoing status quo is implemented and remains in place until the next period. This process continues ad infinitum.

The problem immediately encountered in this framework is that existence results for stationary Markov perfect equilibria provided in the extant game-theoretic literature do not apply. The consequence has been a fast growing body of literature that explicitly constructs stationary Markovian equilibria for bargaining games with an endogenous status quo, and then analyzes the properties of policy outcomes implied by these constructions (e.g., Baron, 1996; Kalandrakis 2004, 2009, 2014; Bowen and Zahran, 2012; Nunnari, 2014; Richter, 2014; Baron and Bowen, 2014; Zápal, 2014; and Anesi and Seidmann, 2015). These analyses are an important development in the study of legislative dynamics; but almost all either assume that the space of alternatives is unidimensional, or focus on pie-division settings where each bargainer's utility only depends on her own share of the pie. There are no known conditions that guarantee the existence of a stationary Markovian equilibrium for more general multidimensional choice spaces.[1]

In this paper, we allow the feasible set of alternatives to be any nonempty subset of multidimensional Euclidean space, and we assume only continuously differentiable, bounded stage utilities. The bargaining protocol is standard, and we permit the voting rule to be any non-collegial rule. We show that when players are sufficiently patient, stationary Markov perfect equilibria in pure strategies can be constructed "close to" any alternative at which the gradients of the players' utilities are linearly independent, i.e., every open neighborhood of an alternative satisfying this condition contains the absorbing points of a stationary Markov perfect equilibrium. In fact, we show that there is a *continuum* of distinct stationary Markov perfect equilibria with absorbing points close to that alternative. Given a set of alternatives of sufficiently high dimension, the linear independence condition holds generically outside a set of alternatives with measure zero, with the implication that equilibria typically abound in such models. In addition, we establish an alternative sufficient condition for existence of a continuum of stationary Markov perfect equilibria that holds in some cases where linear independence of gradients fails everywhere. We emphasize that both conditions are easily verified in economic environments and, together, cover a wide range of applications frequently encountered in this literature. These include pie-division settings and, more generally, the large class of economies with a private good component.

The indeterminacy result extends the approach of Anesi and Seidmann (2015), who

[1]An exception is Duggan and Kalandrakis (2012), who establish existence of stationary Markovian equilibria in pure strategies for general environments. They modify the basic framework by adding noise to the status quo transition and assuming preference shocks in each period. This paper concentrates on existence conditions that do not rely on such noise.

define the concept of simple solution as a list of alternatives for each player and a corresponding list of decisive coalitions such that for each player: the player's utility takes two values over the list of alternatives, a "reward" payoff and a "punishment" payoff; the player is included in and excluded from at least one coalition; and the player receives her reward payoff whenever included in a coalition and receives her punishment payoff whenever excluded. The above authors show that in the pie-division environment, given any simple solution and assuming sufficiently patient players, there is a stationary Markov perfect equilibrium with absorbing set that coincides with the simple solution. We extend the analysis to the general spatial setting by constructing a continuum of simple solutions around any alternative satisfying the linear independence condition, and by making use of the equilibrium construction of Anesi and Seidmann (2015) — who established existence, but not indeterminacy, of Markovian equilibria for the pie-division model. In particular, we are able to shed many of the assumptions usually made in this literature, dropping convexity and compactness of the set of alternatives, and assuming only weak conditions on players' utilities.

Our analysis has important implications for the analysis of legislative policymaking in spatial environments. In high-dimensional policy spaces, assuming discount factors are close to one, constructive techniques that involve an explicit specification of a particular equilibrium must rely on a restrictive selection of equilibria, with the danger that insights derived from those analyses are driven by the equilibrium selection, rather than equilibrium incentives in general. In the absence of further justifications for such a selection, the multiplicity of equilibria we highlight suggest limits on the usefulness of these constructions in predicting the policy outcomes and understanding the dynamics and comparative statics of legislative bargaining. Studies of spatial bargaining with an endogenous status quo thus face an important equilibrium refinement issue.

As mentioned earlier, existence results for stationary Markov perfect equilibria provided in the literature on stochastic games do not apply to the spatial bargaining framework, as they rely on continuity conditions on the transition probability that are violated in the bargaining model (cf. Duggan 2014 for a more detailed discussion). Existence (and characterization) results for Markov perfect equilibria have been obtained in alternative frameworks of dynamic bargaining in which the policy space is finite (Anesi 2010; Diermeier and Fong 2011, 2012; and Battaglini and Palfrey 2012) or without discounting (Anesi and Seidmann 2014) or when the set of possible status quos is countable (Duggan 2014).

In this paper, we restrict attention to spatial bargaining games with non-collegial voting rules. A description of these environments and a formal definition of the equilibrium concept appear in Section 2. Section 3 provides a result on simple solutions that is key for our analysis. Section 4 presents our main result on indeterminacy of equilibria in dynamic bargaining games. Finally, in Section 5, we give formal proofs of our theorems.

2 Spatial Bargaining Framework

In this section, we define the general class of dynamic bargaining games with non-collegial voting rules.

Spatial bargaining games with an endogenous status quo. In each of an infinite number of discrete periods, indexed $t = 1, 2, \ldots$, a finite set of players $N \equiv \{1, \ldots, n\}$, with $n \geq 3$, must reach a collective choice from a nonempty set of alternatives, $X \subseteq \Re^d$, which has full dimension. Let x^t denote the alternative chosen in period t. Bargaining takes place as follows. Each period t begins with a status-quo alternative x^{t-1}, in place from the previous period. Player i is selected with probability $p_i \in (0, 1)$ to propose a policy in X; all players then simultaneously vote to accept or to reject the chosen proposal. It is accepted if a coalition $C \in \mathcal{D}$ of players vote to accept, and it is rejected otherwise, where $\mathcal{D} \subseteq 2^N \setminus \{\emptyset\}$ is the collection of *decisive coalitions,* which have the authority to decide policy in a given period. If proposal y is accepted, then it is implemented in period t and becomes the status quo next period (i.e., $x^t = y$); otherwise the previous status quo, x^{t-1}, is implemented and remains the status quo in period $t + 1$ (i.e., $x^t = x^{t-1}$). This process continues *ad infinitum.* The initial status quo, $x^0 \in X$, is exogenously given.

We assume that $\mathcal{D}$ is nonempty and *monotonic,* i.e., any superset of a decisive coalition is itself decisive: $C \in \mathcal{D}$ and $C \subseteq C'$ imply $C' \in \mathcal{D}$. In addition, we assume that $\mathcal{D}$ is *non-collegial,* in the sense that no player has a veto: we have $N \setminus \{i\} \in \mathcal{D}$ for all $i \in N$. Thus, the model allows for most familiar voting rules, short of unanimity rule (in which case each player has a veto).

The preferences of each player $i \in N$ over X are represented by a continuously differentiable, bounded von Neumann-Morgenstern stage utility function $u_i \colon X \to \Re$. We say u_i is *Euclidean* if there exists $\hat{x}^i \in X$ such that for all $x \in X$, we have $u_i(x) = -||x - \hat{x}^i||^2$. For later use, we say u_i is *pseudo-concave* at x if for all $y \in X$ with $u_i(y) > u_i(x)$, we have $\nabla u_i(x) \cdot (y - x) > 0$. Of course, Euclidean preferences are pseudo-concave. Another example is linear preferences, for which a non-zero gradient $a^i \in \Re^k$ is fixed and for all $x \in X$, we have $u_i(x) = a^i \cdot x$. Given a sequence of alternatives $\{x^t\} \in X^\infty$, player i's payoff is $(1 - \delta_i) \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(x^t)$, where $\delta_i \in [0, 1)$ is her discount factor.

Notable examples. A noteworthy special case of our general environment is that of a *mixed economy,* in which an alternative $x = (x_1, \ldots, x_n, g)$ consists of a private component

$(x_1, \ldots, x_n) \in \Re^n_+$ and possibly a public component $g \in \Re^{d-n}$, $d \geq n$. Here, the set of alternative is $X = \{x : f(-\sum_{i=1}^n x_i, g) \leq 0\}$, where $f\colon \Re^{d-n+1}_+ \to \Re$ is a continuous, strictly monotonic function. We then require that each u_i is strictly increasing function in x_i and constant in $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$; more formally, $\frac{\partial u_i}{\partial x_i}(x) > 0$ and $\frac{\partial u_i}{\partial x_j}(x) = 0$ for all x and all $j \neq i$. We interpret x_i as an amount of a resource allocated to i, and our restriction on utilities reflects the assumption that there are no consumption externalities in the private good. The most obvious example of a mixed economy is, of course, the *pie-division setting,* in which $X = \{(x_1, \ldots, x_n) \in [0,1]^n : \sum_{i=1}^n x_i \leq 1\}$ is the n-dimensional unit simplex; but mixed economies with richer policy spaces are also common in the political-economy literature (e.g., Jackson and Moselle 2002).

It is worth remarking that the alternative version of the pie-division setting, in which the pie must be fully divided, does not constitute a mixed economy, as defined above. This variant, which has received considerable attention in the literature on bargaining (both with and without an endogenous status quo), is also captured by our framework. To this end, define $X = \{(x_1, \ldots, x_{n-1}) \in [0,1]^{n-1} : \sum_{i=1}^{n-1} x_i \leq 1\}$ as the $(n-1)$-dimensional unit simplex, and assume: (i) $\frac{\partial u_i}{\partial x_i}(x) > 0$ and $\frac{\partial u_i}{\partial x_j}(x) = 0$ for all x, all $i < n$ and all $j \in N \setminus \{i, n\}$; and (ii) there exists a continuously differentiable, real-valued function v on $[0,1]$, with $v' > 0$, such that $u_n(x) = v(1 - \sum_{i=1}^{n-1} x_i)$ for all $x = (x_1, \ldots, x_{n-1})$. The set of alternatives, so-defined, has full dimension, and furthermore the players' utilities are pseudo-concave.

Strategies and stationary bargaining equilibrium. We focus on subgame perfect equilibria in which players use stationary Markov (pure) strategies, defined as follows. For any player $i \in N$, a *stationary Markov strategy* $\sigma_i = (\pi_i, \alpha_i)$ consists of a proposal strategy $\pi_i\colon X \to X$, where $\pi_i(x)$ is the proposal made by player i when the current status quo is x (conditional on her being selected to propose), and a voting strategy $\alpha_i\colon X^2 \to \{0,1\}$, where $\alpha_i(x,y)$ is the (degenerate) probability i votes to accept a proposal y when the current default is x. A *stationary Markov perfect equilibrium* is a subgame perfect equilibrium in which all players use stationary Markov strategies.

We follow the standard approach of concentrating throughout on equilibria in stage-undominated voting strategies; i.e., those in which, at any voting stage, no player uses a weakly dominated strategy. Hence, we refer to a pure stationary Markov perfect equilibrium in stage-undominated voting strategies more succinctly as a *stationary bargaining equilibrium.*

Absorbing points and the no-delay property. Every stationary Markov strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ (in conjunction with recognition probabilities) generates a transition function $P^\sigma\colon X^2 \to [0,1]$, where $P^\sigma(x,y)$ is the probability, given σ, that the alternative implemented in the next period is y, given that the alternative implemented in the current period is x.[2] We say that $x \in X$ is an *absorbing point* of σ if and only if $P^\sigma(x,x) = 1$, and we denote the set of absorbing points of σ by $A(\sigma) \equiv \{x \in X : P^\sigma(x,x) = 1\}$. We will say that σ is *no-delay* if and only if: (i) $A(\sigma) \neq \emptyset$; and (ii) for all $x \in X$, there is $y \in A(\sigma)$ such that $P^\sigma(x,y) = 1$. In words, a strategy profile is no-delay if an absorbing point is implemented in every period (both on and off the equilibrium path).

3 Continuum of Simple Solutions

In this section, we make some observations about simple solutions and stationary bargaining equilibria of spatial bargaining games.

Preliminaries on simple solutions and existence of equilibria. The next definition extends Anesi and Seidmann's (2015) definition of a simple solution for the pie-division setting to the general spatial setting.

Definition 1. *Let $\mathcal{C} \equiv (C_i)_{i\in N} \in \mathcal{D}^n$ be an ordered n-tuple of decisive coalitions such that for each $i \in N$, we have $i \in C_i$ and, for some $j \in N \setminus \{i\}$, $i \notin C_j$. An ordered n-tuple of alternatives $\bar{s} = (\bar{x}^1, \ldots, \bar{x}^n) \in X^n$ is a $\mathcal{C}$-**simple solution** if there exist $(v_1, \ldots, v_n), (w_1, \ldots, w_n) \in \Re^n$ such that*

(i) for all $i \in N$, $v_i > w_i$, and

(ii) for all $i, j \in N$,

$$u_j(\bar{x}^i) = \begin{cases} v_j & \text{if } j \in C_i, \\ w_j & \text{otherwise.} \end{cases}$$

*Then $\bar{s} \in X^n$ is a **simple solution** if there exists an n-tuple $\mathcal{C}$ of decisive coalitions (as defined above) such that $\bar{s}$ is a $\mathcal{C}$-simple solution.*

Why study simple solutions? Anesi and Seidmann (2015) show that in pie-division settings, simple solutions identify alternatives that are absorbing points of stationary bargaining equilibria for the corresponding bargaining games. Simple solutions for the general spatial setting are justified on precisely the same grounds.[3]

Proposition 1. *Let $\bar{s} = (\bar{x}^1, \ldots, \bar{x}^n)$ be a simple solution. There is a threshold $\bar{\delta} \in (0,1)$ such that if $\min_{i\in N} \delta_i > \bar{\delta}$, then there exists a no-delay stationary bargaining equilibrium σ*

[2] As all players use pure strategies, $P^\sigma(x,\cdot)$ is a discrete probability density with $|\text{supp}(P^\sigma(x,\cdot))| \leq n$ for all $x \in X$.

[3] The proof of Proposition 1 exactly parallels that of Theorem 1 in Anesi and Seidmann (2015) and hence is omitted.

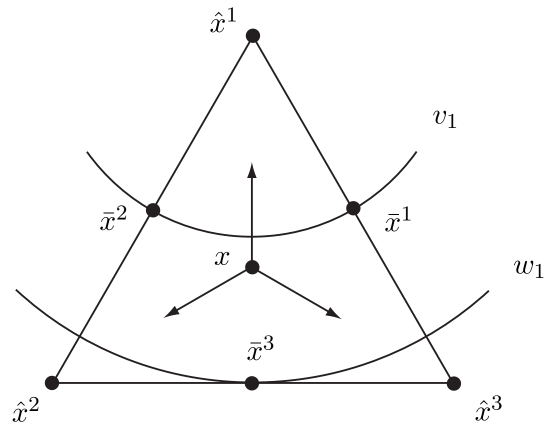

Figure 1: Simple solution under (C2)

with absorbing points $\{\bar{x}^1, \ldots, \bar{x}^n\}$*. Moreover, this threshold can be specified as a continuous function* $\bar{\delta}(\bar{x}^1, \ldots, \bar{x}^n)$ *of the simple solution.*

Therefore, if one can show that a simple solution exists, then necessarily the spatial bargaining game (with sufficiently large discount factors) will possess a stationary bargaining equilibrium.

Multiplicity of simple solutions. To use Proposition 1 for an equilibrium existence result, it remains to find conditions under which a simple solution exists. Two such conditions take the form of restrictions on the gradients of players' utility functions at interior points of X.

(C1) $\{\nabla u_i(x) : i \in N\}$ is linearly independent.

Condition (C1) implies the intuitive property that we can obtain values of the utility profile $u = (u_1, \ldots, u_n)$ in an open neighborhood of $u(x) \in \Re^n$ by arbitrarily small variations of x, i.e., the Jacobian of u at x has full row rank. It is thus relatively easy to verify in economic applications. For instance, it applies whenever the set of alternatives has a private good component: given our formulation of mixed economies, (C1) holds at any alternative x in the interior of X. In particular, this holds in the widely studied pie-division setting. More generally, in multidimensional settings such that $d \geq n$, the condition generically holds outside a closed set of measure zero of alternatives (Smale 1974). Note, however, that it is violated at Pareto optimal alternatives, and thus it is violated in the version of pie-division in which the entire pie must be consumed.

Simple solutions may exist even in cases where no alternative satisfies (C1), as when $d < n$. To see this, let $N = \{1, 2, 3\}$ and assume each $i \in N$ has Euclidean preferences on $X \subseteq \Re^2$ with equidistant ideal alternatives $\hat{x}^i$, $i = 1, 2, 3$. As is easily seen in Figure 1 by setting $C_1 = \{1, 3\}$, $C_2 = \{1, 2\}$, and $C_3 = \{2, 3\}$, simple solutions such as $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ can be constructed in any open neighborhood of alternative x, which lies at the center of the convex hull of the players' ideal alternatives. Though (C1) is violated everywhere in X, x satisfies an alternative sufficient condition for existence of a simple solution, defined next.

(C2) (i) There exist coefficients $\alpha_1, \ldots, \alpha_n > 0$ such that $\sum_{i \in N} \alpha_i \nabla u_i(x) = 0$; (ii) for all $i \in N$, u_i is pseudo-concave at x; and (iii) for all $i \in N$, $\{\nabla u_j(x) : j \in N \setminus \{i\}\}$ is linearly independent.

In contrast to (C1), condition (C2) holds at all interior alternatives in the version of pie-division with full consumption of the pie. This and the previous examples show that conditions (C1) and (C2) are easy to check and apply to a range of economic environments of interest. A key result of this paper establishes existence of a continuum of simple solutions in non-collegial, spatial bargaining games under either condition.

Theorem 1. *Let x be any interior point of X at which either (C1) or (C2) is satisfied. Every open neighborhood U of x is such that U^n contains a continuum of simple solutions.*

Outline of the argument. A detailed proof of Theorem 1 is provided in Section 5. To see the idea behind that proof, consider an alternative x that fulfills (C1) and, for expositional simplicity, suppose that $N = \{1, 2, 3\}$.[4] Our first step is to find a triple of alternatives (x^1, x^2, x^3) such that each player i's utility can only take two possible values over the three alternatives: her "reward payoff" v_i or her "punishment payoff" w_i. To this end, define f as the function that maps vectors of alternatives $(x^1, x^2, x^3) \in X^3$ to corresponding utility vectors $(u_i(x^j))_{i,j \in N} \in \Re^9$. The argument is depicted in Figure 2, where we place $(u_1(x), u_2(x), u_3(x))$ at the center of the simplex in $\Re^3$. Condition (C1) implies that the Jacobian of f has full row rank at x. By the local submersion theorem (e.g., Guillemin and Pollack, 1974), therefore, we can perturb x to alternatives x^1, x^2, x^3, so as to give each player i her "punishment payoff" at x^i while giving the other players their "reward payoffs," e.g., for sufficiently small $\epsilon > 0$, we can set $u_i(x^i) = u_i(x) - 2\epsilon \equiv w_i$ and $u_j(x^i) = u_j(x) + \epsilon \equiv v_j$ for all i and $j \neq i$.

To apply Definition 1, we simply let $C_1 = \{1, 3\}$, $C_2 = \{1, 2\}$, $C_3 = \{2, 3\}$, $\bar{x}^1 = x^2$, $\bar{x}^2 = x^3$, and $\bar{x}^3 = x^1$. As $\mathcal{D}$ is non-collegial, coalitions C_1, C_2 and C_3 must all be decisive.

[4] An analogous proof strategy can be used for the case where x satisfies condition (C2) instead.

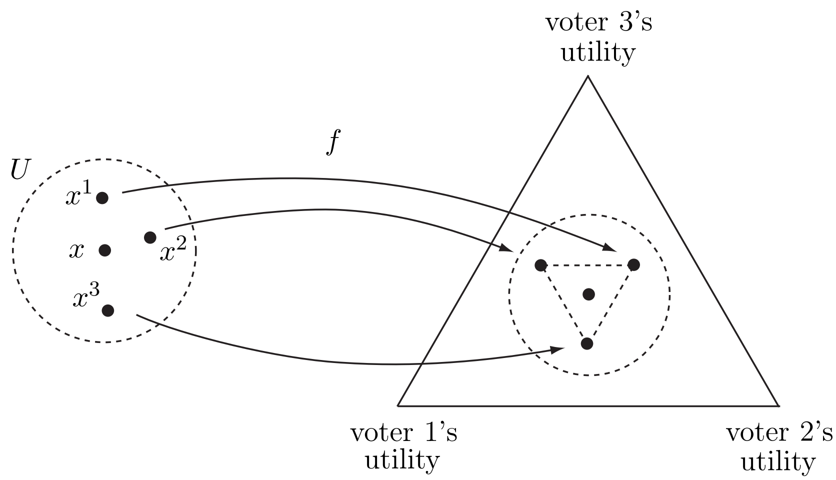

Figure 2: Mapping to utility vectors

Moreover, we have

$$u_j(\bar{x}^i) = \begin{cases} u_j(x) + \epsilon & \text{if } j \in C_i \ , \\ u_j(x) - 2\epsilon & \text{otherwise,} \end{cases}$$

for all $i, j \in N$. Hence, $(x^1, x^2, x^3) = (\bar{x}^3, \bar{x}^1, \bar{x}^2)$ constitutes a simple solution. It is readily checked that we can use the same argument for a continuum of values of ϵ that each yield a different simple solution (cf. Section 5).

4 Indeterminacy of Stationary Bargaining Equilibria

Combined, Proposition 1 and Theorem 1 immediately yield an equilibrium existence result for the spatial bargaining game when the dimension of the set of alternatives is high: as discount factors become close to one, absorbing points of stationary bargaining equilibria exist "around" every alternative that satisfies either (C1) or (C2). More significant are the implications of Theorem 1 for the predictive power of stationary bargaining equilibria in this class of games: when players are sufficiently patient, the spatial bargaining game admits a continuum of such equilibria. Our main result, next, states this formally and establishes indeterminacy of stationary bargaining equilibria.

Theorem 2. *Let x be any interior point of X at which either (C1) or (C2) is satisfied. For every open neighborhood U of x, there exists $\bar{\delta} \in (0,1)$ such that if $\min_{i \in N} \delta_i > \bar{\delta}$, then there is a continuum of simple solutions in U^n corresponding to absorbing sets of no-delay stationary bargaining equilibria with discount factors $\delta_1, \ldots, \delta_n$.*

The political-economy literature on bargaining games with an endogenous status quo has devoted considerable attention to the set A^* of *dynamically stable* alternatives, i.e., the alternatives that can be supported as long run outcomes of stationary bargaining equilibria. Formally, A^* consists of every alternative x such that there exists $\bar{\delta} \in (0,1)$ such that if $\min_{i \in N} \delta_i > \bar{\delta}$, then there is a stationary bargaining equilibrium σ for discount factors $\delta_1, \ldots, \delta_n$ such that $x \in A(\sigma)$. In terms of predicting bargaining outcomes, however, the characterization of dynamically stable alternatives is only informative if A^* is "small" relative to the set of alternatives. This is typically not the case in spatial bargaining games with high dimensional spaces. It follows as a corollary of Theorem 2 that under generic conditions, when players are sufficiently patient, the dynamically stable alternatives are dense in the set of alternatives.

Corollary 1. *Assume that the set of alternatives at which (C1) or (C2) hold is dense in* $\text{int}X$*. The set A^* of dynamically stable alternatives given $\delta_1, \ldots, \delta_n$ is dense in* $\text{int}X$*.*

The above observation is reminiscent of the cycling results in the social choice literature (e.g., McKelvey, 1979). Just as the top cycle is generically dense in the set of alternatives in sufficiently high dimensional spaces, we find that long-run bargaining outcomes for any such environment are highly indeterminate. Whereas McKelvey's chaos theorem evokes the picture of collective choices moving arbitrarily through the set of alternatives over time, our results establish the possibility that collective choices via dynamic bargaining can come to rest at arbitrary locations in the set of alternatives. Although the nature of the indeterminacy is different, the results appear to present similar difficulties for the prediction and analysis of social choices in dynamic environments.

5 Proofs of Theorems

Proof of Theorem 1 Let x be an interior point of X that satisfies either (C1) or (C2), and let $U \subseteq X$ be an open neighborhood of x. The proof of Theorem 1 relies on the following simple observation. If the vector of alternatives $(x^1, \ldots, x^n) \in X^n$ satisfies $u_i(x^i) \leq u_i(x) < u_i(x^j) = u_i(x^k)$ for every selection of three distinct players $i, j, k \in N$, then $(x^1, \ldots, x^n)$ constitutes a simple solution. To see this, set $C_i = N \setminus \{i+1\}$, set $\bar{x}^i = x^{i+1}$, set $v_i = u_i(x^{i+1})$, and set $w_i = u_i(x^i)$ for all $i \neq n$; and set $C_n = N \setminus \{1\}$, set $\bar{x}^n = x^1$, set $v_n = u_n(x^1)$, and set $w_n = u_n(x^n)$. As $\mathcal{D}$ is non-collegial, we have $(C_i)_{i \in N} \in \mathcal{D}^n$. It immediately follows from Definition 1 that $(x^1, \ldots, x^n)$ constitutes a simple solution.

To establish the theorem, therefore, it remains to show that there is a continuum of such alternative vectors in U^n.

Case 1: x satisfies (C1). That is, $\{u_i(x) : i \in N\}$ is linearly independent. Define the mapping $f\colon X^n \to \Re^{n^2}$ by

$$f(x^1, \ldots, x^n) = \begin{pmatrix} u_1(x^1) \\ \vdots \\ u_1(x^n) \\ \vdots \\ u_n(x^1) \\ \vdots \\ u_n(x^n) \end{pmatrix}.$$

The derivative of f at arbitrary $(x^1, \ldots, x^n) \in X^n$ is the $n^2 \times nd$ matrix

$$Df(x^1, \ldots, x^n) = \begin{bmatrix} Du_1(x^1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & Du_1(x^n) \\ Du_2(x^1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & Du_2(x^n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Du_n(x^1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & Du_n(x^n) \end{bmatrix},$$

where we view $Du_i(x^j)$ as a $1 \times d$ row matrix. By assumption, this matrix has full row rank at $(x, \ldots, x)$. Moreover, we have $d \geq n$, since the individuals' gradients are linearly independent, and therefore $dn \geq n^2$.

Let $y = (y^1, \ldots, y^n) = f(x, \ldots, x)$, where $y^i = (u_i(x), \ldots, u_i(x))$ is then the n-fold copy of individual i's utility from x. By the local submersion theorem (e.g., Guillemin and Pollack, 1974), we can choose an arbitrarily small open set $\tilde{U} \subseteq U^n$ containing $(x, \ldots, x)$ such that the image $\tilde{V} \equiv f(\tilde{U})$ is an open set containing y. Therefore, there exists $\epsilon > 0$

such that

$$z_\epsilon = \begin{pmatrix} u_1(x)+\epsilon \\ \vdots \\ u_1(x)+\epsilon \\ u_1(x)-(n-1)\epsilon \\ \\ u_2(x)-(n-1)\epsilon \\ u_2(x)+\epsilon \\ \vdots \\ u_2(x)+\epsilon \\ \\ \vdots \\ \\ u_n(x)+\epsilon \\ \vdots \\ u_n(x)+\epsilon \\ u_n(x)-(n-1)\epsilon \\ u_n(x)+\epsilon \end{pmatrix}$$

belongs to $\tilde{V}$.

Since $z_\epsilon \in \tilde{V}$, there is a vector $(x^1_\epsilon, \ldots, x^n_\epsilon) \in U^n$ such that $f(x^1_\epsilon, \ldots, x^n_\epsilon) = z_\epsilon$, and $(x^1_\epsilon, \ldots, x^n_\epsilon)$ constitutes a simple solution. We can similarly construct vectors $z_\gamma \in V$ and $(x^1_\gamma, \ldots, x^n_\gamma) \in U^n$ for all $\gamma \in (0, \epsilon)$. By construction, $\gamma_1 \neq \gamma_2$ implies $z_{\gamma_1} \neq z_{\gamma_2}$ and, therefore, $(x^1_{\gamma_1}, \ldots, x^n_{\gamma_1}) \neq (x^1_{\gamma_2}, \ldots, x^n_{\gamma_2})$. We conclude that there is a continuum of simple solutions contained in U^n.

Case 2: x satisfies (C2). That is, (i) there exist coefficients $\alpha_1, \ldots, \alpha_n > 0$ such that $\sum_i \alpha_i \nabla u_i(x) = 0$, (ii) for all individuals i, u_i is pseudo-concave at x, and (iii) for all i,

$\{\nabla u_j(x) : j \neq i\}$ is linearly independent. Define the mapping $f\colon X^n \to \Re^{n(n-1)}$ by

$$f(x^1, \ldots, x^n) \;=\; \begin{pmatrix} u_1(x^2) \\ \vdots \\ u_1(x^n) \\ \\ u_2(x^1) \\ u_2(x^3) \\ \vdots \\ u_2(x^n) \\ \\ u_n(x^1) \\ \vdots \\ u_n(x^{n-1}) \end{pmatrix}.$$

The derivative of f at arbitrary $(x^1, \ldots, x^n) \in X^n$ is the $n(n-1) \times nd$ matrix

$$Df(x^1, \ldots, x^n) \;=\; \begin{bmatrix} 0 & \nabla u_1(x^2) & 0 & \cdots & 0 \\ 0 & 0 & \nabla u_1(x^3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nabla u_1(x^n) \\ \nabla u_2(x^1) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \nabla u_2(x^3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nabla u_2(x^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nabla u_n(x^1) & 0 & 0 & \cdots & 0 \\ 0 & \nabla u_n(x^2) & 0 & \cdots & 0 \\ 0 & 0 & \nabla u_n(x^3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

By (iii), this matrix has full row rank at $(x, \ldots, x)$. Moreover, (iii) implies that $d \geq n-1$, and therefore $nd \geq n(n-1)$.

Let $y = (y^1, \ldots, y^n) = f(x, \ldots, x)$, where now y^i is an $(n-1)$-dimensional vector with $u_i(x)$ in each coordinate. By the local submersion theorem, there is an arbitrarily small open set $\tilde{U} \subseteq U^n$ containing $(x, \ldots, x)$ such that the image $\tilde{V} = f(\tilde{U})$ is an open set containing y. Therefore, there exists $\epsilon > 0$ such that

$$z_\epsilon = \begin{pmatrix} u_1(x)+\epsilon \\ \vdots \\ u_1(x)+\epsilon \\ \\ u_2(x)+\epsilon \\ \vdots \\ u_2(x)+\epsilon \\ \\ \vdots \\ \\ u_n(x)+\epsilon \\ \vdots \\ u_n(x)+\epsilon \end{pmatrix}$$

belongs to $\tilde{V}$. That is, in contrast to the first case, we simply modify y by adding ϵ to each component. Since $z_\epsilon \in \tilde{V}$, there exists $(x^1_\epsilon, \ldots, x^n_\epsilon) \in U^n$ such that $f(x^1_\epsilon, \ldots, x^n_\epsilon) = z_\epsilon$.

To see that $(x^1_\epsilon, \ldots, x^n_\epsilon)$ constitutes a simple solution, note that for each i and each $j \neq i$, we have $u_i(x^j_\epsilon) = u_i(x)+\epsilon$, so the "reward payoffs" are the same across x^j_ϵ, $j \neq i$, for each individual i. We now claim that for each i, we have $u_i(x^i_\epsilon) < u_i(x)$, so that x^i_ϵ yields the "punishment payoff" for i. Indeed, for each $j \neq i$, $u_j(x^i_\epsilon) > u_j(x)$ and pseudo-concavity, from (ii), imply $\nabla u_j(x) \cdot (x^i_\epsilon - x) > 0$. Then (i) implies

$$\nabla u_i(x) \cdot (x^i_\epsilon - x) = \sum_{j \neq i} -\frac{\alpha_j}{\alpha_i} \nabla u_j(x) \cdot (x^i_\epsilon - x) < 0.$$

Finally, pseudo-concavity then implies $u_i(x^i_\epsilon) \leq u_i(x) < u_i(x)+\epsilon$, as desired.

By same argument as in Case 1, this in turn implies that there is a continuum of simple solutions, completing the proof of Theorem 1.

Proof of Theorem 2 In the proof of Theorem 1, we define a mapping $f\colon X^n \to \Re^{n^2}$ and show that the Jacobian $Df(x,\ldots,x)$ has full row rank for every interior x satisfying (C1) or (C2). Thus, the column rank of $Df(x,\ldots,x)$ is n^2, and there is a subspace $L \subseteq \Re^{kn}$ with dimension n^2 such that the linear transformation from L to $\Re^{n^2}$ given by the matrix $Df(x,\ldots,x)$ is bijective. Let

$$W = ((x,\ldots,x)+L) \cap X^n$$

be the n^2-dimensional manifold defined, essentially, by translating the linear subspace L to $(x, \ldots, x)$. define $\tilde{f} \colon W \to \Re^{n^2}$ as the restriction of f to W, and note that the derivative $D\tilde{f}$ is invertible. Now define the mapping $g \colon W \times \Re \to \Re^{n^2}$ by

$$g(w, \gamma) \quad = \quad \tilde{f}(x^1, \ldots, x^n) - z_\gamma,$$

where we write $w = (x^1, \ldots, x^n)$, and note that this mapping is continuously differentiable. Moreover, $g(x, \ldots, x, 0) = 0$, and if x satisfies (C1) or (C2), then the derivative of g with respect to w is invertible. Then the implicit function theorem (e.g., Theorem 11.2 of Loomis and Sternberg, 1968) implies that there exist $\epsilon > 0$ and a continuous function $\xi \colon (0, \epsilon) \to W$ such that for all $\gamma \in (0, \epsilon)$, we have $g(\xi(\gamma), \gamma) = 0$. Setting $(x^1_\gamma, \ldots, x^n_\gamma)$ equal to $\xi(\gamma)$ for all $\gamma \in (0, \epsilon)$, we obtain the simple solution $(x^1_\gamma, \ldots, x^n_\gamma)$ as a continuous function of γ.

Now, to prove Theorem 2, consider any alternative $x \in \mathrm{int}X$ such that (C1) or (C2) hold, and consider any open neighborhood $U \subseteq X$ of x. In the proof of Theorem 1, modified as above, we obtain $\epsilon > 0$ and a set $S_\epsilon = \{\bar{s}_\gamma : \gamma \in (0, \epsilon)\}$ of distinct simple solutions with $S_\epsilon \subseteq U^n$ and such that $\bar{s}_\gamma = (x^1_\gamma, \ldots, x^n_\gamma)$ is continuous as a function of γ on $(0, \epsilon)$. From Proposition 1, it follows that for each $\gamma \in (0, \epsilon)$, there is a threshold $\bar{\delta}(x^1_\gamma, \ldots, x^n_\gamma) \in (0, 1)$ such that if $\min_{i \in N} \delta_i > \bar{\delta}(x^1_\gamma, \ldots, x^n_\gamma)$, then there exists a no-delay stationary bargaining equilibrium with absorbing points $\{x^1_\gamma, \ldots, x^n_\gamma\}$. Moreover, Proposition 1 establishes continuity of the threshold $\bar{\delta}$ as a function of the simple solution, and it follows that $\bar{\delta}(x^1_\gamma, \ldots, x^n_\gamma)$ is continuous as a function of γ on $(0, \epsilon)$. Thus, it takes a maximum, say $\hat{\delta}$, on the interval $[\frac{\epsilon}{2}, \frac{2\epsilon}{3}]$. We conclude that if $\min_{i \in N} \delta_i > \hat{\delta}$, it follows that for each simple solution in the set $\{\bar{s}_\gamma : \gamma \in [\frac{\epsilon}{2}, \frac{2\epsilon}{3}]\}$, there is a no-delay stationary bargaining equilibrium with absorbing points $\{x^1_\gamma, \ldots, x^n_\gamma\}$. As this set has the cardinality of the continuum and $\hat{\delta} < 1$, this completes the proof of Theorem 2.

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