Over-the-Counter Markets with Bargaining Delays: The Role of Public Information in Market Liquidity

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Abstract

In many over-the-counter (OTC) asset markets, price is negotiated bilaterally and the bargaining over price takes time. This paper develops a dynamic equilibrium model of OTC markets with both search delays and endogenous bargaining delays arising from the lack of public information about assets. We first show that bargaining delays arise even when agents have precise information about the asset quality, as long as the public information is crude. We derive implications of both trade delays for prices and liquidity that differ from that in the complete- and asymmetric- information models. Conditional on public information, the liquidity is U-shaped in the quality and assets in the middle of the quality range may not be traded. Search and bargaining delays have opposite effects on the range of traded assets showing that the reduction in search delays through greater transparency improves risk-sharing but hurts liquidity.

Keywords: *search friction, bargaining delay, liquidity, over-the-counter markets, transparency, private and public information*

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1 Introduction

Many important asset markets are decentralized. Examples include over-the-counter (OTC) markets for commercial and residential real estate, asset-backed securities, derivatives, corporate and municipal bonds, credit-default swaps, private equity, sovereign debt, bank loans, etc. In such markets, prices are negotiated bilaterally and it takes parties time to agree on the price. These bargaining delays can range from months as in the real estate or private equity to hours or even minutes as in the most liquid parts of the bond market. The existing literature pioneered by Duffie, Gârleanu and Pedersen (2005) adopts the generalized Nash bargaining solution and thus abstracts from bargaining delays focusing instead on search delays. In fact, search delays are thought of as a reduced form for all types of trade delays.[1] The goal of this paper is to understand how justified is this approach and to disentangle the effect of two forms of trade delay on the asset liquidity and pricing.

The novelty of our approach is in the departure from the generalized Nash bargaining solution. The key theoretical observation is that in the standard alternating-offer bargaining game (see Binmore et al. (1986)), bargaining delays arise even when parties have very precise private signals about the asset quality, as long as the public information is crude. We introduce the *screening bargaining solution* which captures in the reduced form bargaining delays arising in the almost public information limit, as the precision of the private signals about the quality goes to infinity while holding fixed the precision of the public information.[2] Despite very precise private signals, parties are not able to credibly reveal their private information to the opponent other than through trade delay. In the almost public information limit, bargaining delays are determined by the precision of public information about the asset, rather than parties' private information.

This gap between public and private information is relevant in many OTC markets where only a limited amount of public information about assets is available in the form of credit ratings, past quotes, etc., while investors' information is more precise due to their private sources and expertise in evaluating the assets.[3] Moreover, the connection

[1]Duffie (2012) summarizes the current approach: "[s]earch delays ... proxy for delays associated with reaching an awareness of trading opportunities, arranging financing and meeting suitable legal restrictions, negotiating trades, executing trades, and so on."

[2]This terminology comes from the epistemic literature (Aumann (1999)) where the public information establishes the common knowledge among investors. As the private signals of parties becomes more precise, values become almost common knowledge, thus, the term almost public information limit.

[3]The Committee on the Global Financial System (2005) gives the following account of the OTC trade: "Interviews with large institutional investors in structured finance instruments suggest that they do not rely on ratings as the sole source of information for their investment decisions ... Indeed, the relatively

between the quality of public information and liquidity is important for understanding the effect of heightened market uncertainty or greater transparency on the functioning of such markets.

We incorporate bargaining delays arising in the limit of almost public information into otherwise standard dynamic equilibrium model of OTC markets à la Duffie et al. (2007). This approach makes the analysis of the OTC model particularly tractable, as it abstracts from learning about the quality of assets.[4] Specifically, we consider an economy in which investors are occasionally hit by liquidity shocks, and to share risks, they can trade a continuum of assets in the market with search and bargaining delays. The bargaining delays are captured by the screening bargaining solution and determined in equilibrium by investors' valuations which in turn depend on their ability to sell quickly assets in the future as well as the steady-state distribution of assets among investors in the economy. We solve for the unique steady-state equilibrium of this economy and derive implications of bargaining delays for asset prices and standard liquidity measures, such as bid-ask spread, trade volume, turnover, trade delays, zero-trading days.

Our main finding is that the liquidity implications of bargaining delays arising from the lack of public information about assets are drastically different from that in the adverse selection models (e.g. Guerrieri and Shimer (2014)) or search-and-bargaining models of liquidity (e.g. Duffie et al. (2005)). Conditional on the public information about the asset quality, the asset trade volume and turnover are U-shaped in the quality, which is in contrast to the adverse selection models of asset liquidity predicting decreasing realtionship (see e.g. Guerrieri and Shimer (2014))). This pattern arises from the dynamics of the negotiation when both sides have very precise information about the quality but lack common knowledge about the quality. The buyer of a high quality asset and the seller of a low quality asset are willing to accept early on an offer close to maximal and minimal prices, resp., while buyers and owners of assets in the middle of the quality range prefer to delay trade to hold out for a more favorable price offer.

Bargaining delays operate quite differently from random search delays used in the search-and-bargaining models of OTC markets. Bargaining delays endogenously vary across asset qualities and this endogenous heterogeneity of assets creates an extensive

coarse filter a summary rating provides is seen, by some, as an opportunity to trade finer distinctions of risk within a given rating band. Nevertheless, rating agency 'approval' still appears to determine the marketability of a given structure to a wider market."

[4]This can also be a realistic description of some OTC markets. For example, Pérignon et al. (2015) find that in the the European wholesale funding market the asymmetric information does not have a first-order effect on the allocation of funds.

trade margin: the range of asset qualities may not be traded in equilibrium. When the buyer finds the seller, the buyer compares the bargaining delay associated with the seller's asset with the outside option of searching further. For assets that involve lengthy negotiation, the buyer prefers to continue the search, while she accepts assets with sufficiently short negotiation times. We conclude that the current approach that focuses only on search delay is with a loss as it ignores the extensive trade margin. The extensive margin can be more important for liquidity than the intensive margin (the length of trade delay). For example, when search delays for buyers are very short, buyers accept for trade only the most quickly negotiated assets. As a result, the market is illiquid as many sellers cannot liquidate their positions, while by looking at the intensive margin the market may seem very liquid as *observed* search and bargaining delays are short.

We further show that search and bargaining delays have opposite effects on the extensive margin. Higher bargaining delays reduce the set of traded assets. For example, when the public information deteriorates, bargaining delays increase and buyers prefer to continue the search for a wider range of assets. This is consistent with the dried-up liquidity during periods of heightened market uncertainty when infrequently updated credit ratings become less reliable in assessing the risks associated with the asset.[5] Perhaps surprisingly, an increase in search delays expands the set of traded assets. When it is harder to find a trade partner, investors' outside option of searching for an alternative partner deteriorates which makes buyers willing to accept a wider range of assets for trade.

The effect of two frictions on the market liquidity has an important implication for the design of OTC markets. On the one hand, better quality of credit ratings, standardization of products, benchmarks improve the quality of public information of assets and thus reduce bargaining delays. On the other hand, electronic trade platforms or active brokerage improve the search. Our analysis implies that the former policies expand the set of actively traded assets, while the latter reduce the market liquidity. Thus, making the search more efficient can improve the risk-sharing through shorter search times, but the improvement need not be Pareto as less sellers are able to liquidate their positions. This finding is consistent with the existing mixed evidence on the effect on liquidity of the post-trade transparency in the corporate bonds market (see Bessembinder et al. (2006), Edwards et al. (2007), Goldstein et al. (2007), Asquith et al. (2013)), as the post-trade transparency improves both the quality of public information and the search efficiency.

In the analysis of liquidity, different assets act as substitutes for risk-sharing. In the

[5] In the recent financial crisis, the significant increase in downgrades of financial products (see Benmelech and Dlugosz (2010), Ashcraft et al. (2010)) indicates the drop in the accuracy of credit ratings.

recent financial crisis of 2007-2008, traders reacted to the increase in market uncertainty by a shift in their preferences towards safer and more liquid assets, a phenomenon known as flight-to-liquidity (Dick-Nielsen et al. (2012), Friewald et al. (2012)). Similarly, opponents of greater transparency in OTC markets point out that it causes migration of trade to certain asset classes hurting the liquidity of the market as a whole. We extend the baseline model to take into account the substitutability between asset classes. An increase in the bargaining friction for one asset class can result in flight-to-liquidity episodes wherein investors migrate to trading assets with lower bargaining delays, which exacerbates the negative effect of the increased market uncertainty on the liquidity. Interestingly, once we take into account the asset substitutability, even the reduction in bargaining delays can have adverse effects. If the reduction is uneven across asset classes, and as a result, there is an asset class that is significantly more liquid than the rest of the market, then investors will migrate to trading assets in this class. This adversely affects the liquidity of the rest of the market and can result in an overall decrease in the market liquidity. This reveals the negative effects of gradual transparency policies. For example, the recent introduction of mandatory trade reporting in the corporate bonds market was introduced in several phases. Asquith et al. (2013) shows that this hurt mostly the trade volumes of high-yield bonds for which the post-trade transparency was introduced later than for the investment grade bonds.

Related literature This paper is related to several strands of literature. First, it contributes to the growing literature on the search and bargaining models of OTC markets pioneered by Duffie et al. (2005) and further developed to account for risk-aversion (Duffie et al. (2007)), unrestricted asset holdings (Lagos and Rocheteau (2007, 2009)), asset heterogeneity (Vayanos and Weill (2008), Weill (2008)), agent heterogeneity (Vayanos and Wang (2007), Shen et al. (2015), Hugonnier et al. (2014), Üslü (2015)). This literature abstracts from bargaining delays by adopting the generalized Nash bargaining solution which is a reduced form for bargaining with complete information.[6] To the best of our knowledge, this paper is the first to study the implications of microfounded bargaining delays for OTC liquidity and show that bargaining delays operate quite differently from

[6]Similarly, the generalized Nash bargaining solution has been used in the monetary search literature (e.g. Trejos and Wright (1995)) and labor search literature (e.g. Hosios (1990)).

search delays.[7,8]

Second, our microfoundations for the screening bargaining solution contribute to the bargaining literature with two-sided private information about values (Ausubel and Deneckere (1992), Cho (1990), Cramton (1984), Ausubel and Deneckere (1993)), which focuses exclusively on the case of independent values,[9] and the literature on bargaining with interdependent values, but one-sided private information (Deneckere and Liang (2006), Fuchs and Skrzypacz (2013), Gerardi et al. (2014)). In contrast, we study bargaining with two-sided private information about correlated (through the asset quality) values and show that bargaining delays arise even when offers are frequent and the correlation is almost perfect (in fact, delay is necessary to attain the Nash split in the limit). Larsen (2013) analyzes the bargaining model in which values are correlated through the publicly observed information about the object and the two-sided private information is about idiosyncratic components of values (thus, the model essentially reduces to that with two-sided independent private information).

Third, our paper offers an information-based theory of liquidity that builds on the lack of public information about assets rather than the asymmetric information. Dynamic asset trading models with adverse selection (Guerrieri and Shimer (2014), Kurlat (2013), Chang (2014)) predict a decreasing relationship: in order to provide incentives for sellers of lower-quality assets to reveal their quality, such assets should be more liquid.[10] The contrast, we predict a U-shaped dependence of the liquidity on the asset quality.[11] The information asymmetry models are most relevant in the primary markets (e.g. originator of mortgages v.s. investors buying MBSs), while our model better captures the gap between public

[7] If anythings, the existing results suggest that the search and bargaining frictions are similar, e.g. Lagos and Rocheteau (2009) show that the bargaining power of market makers operates similarly to the search friction.

[8] More broadly, the paper is related to the literature on asset pricing with transaction costs which explored exogenous proportional transaction costs (Constantinides (1986), Heaton and Lucas (1996), Huang (2003)), fixed trading costs (Lo et al. (2004)) and exogenous bid-ask spreads (Amihud and Mendelson (1986)). Like Duffie et al. (2005), this paper focuses on a different type of transaction costs, the opportunity costs of delayed trade, however, in our model the delay, rather than being exogenously given, is endogenously determined via the amount of public information about assets.

[9] Independence assumption is also common in bargaining with two-sided private information about discount factors (Watson (1998)) and players' rationality (see Abreu and Gul (2000), Kambe (1999) among others).

[10] There is also a growing literature that introduces the adverse selection into the Walrasian competitive equilibrium (e.g. see Guerrieri et al. (2010), Kurlat (2016)) and imperfectly competitive equilibrium (e.g. see Lester et al. (2015)).

[11] In this respect, our paper is related to He and Milbradt (2014), Chen et al. (2014) analyzing the feedback loop between the liquidity and default, and showing that assets closer to default are associated with higher bid-ask spreads. Both their channel and prediction are different from ours.

information and private information of sophisticated investors in secondary markets (e.g. two hedge funds dealing corporate bonds).

Forth, our paper contributes to the theoretical literature that studies the effect of transparency on the efficiency and liquidity of OTC markets (Duffie et al. (2015), Asriyan et al. (2015)). This literature shows that higher transparency reduces the information asymmetry between agents, and hence, may lead to more efficient risk sharing and higher liquidity. Our paper shows that the effect of transparency on liquidity is ambiguous depending on whether it leads to the reduction in the bargaining or search friction. It also shows that adverse effects can arise because of the asset substitutability.

Fifth, the paper is related to the theoretical literature on search-and-bargaining pioneered by Rubinstein and Wolinsky (1985) most of which focuses on the case of complete information and hence immediate agreement (see Osborne and Rubinstein (1990), Gale (2000) for an excellent survey). Exceptions include work by Satterthwaite and Shneyerov (2007) and Lauermann and Wolinsky (2014) who study the conditions for convergence to the Walrasian outcomes in search models with incomplete information where allocations are determined by static auction mechanisms. In contrast, our focus is on bargaining delays, and because of the bargaining friction, our model does not converge to the competitive outcome even as the search friction vanishes. Another paper that explicitly incorporates bargaining delays into a search market is Atakan and Ekmekci (2014). In their model, investors can imitate exogenously given commitment types requesting a fixed share of the surplus, while in our model all investors are rational.

The paper is organized as follows. Section 2 presents the OTC model. Section 3 introduces the screening bargaining solution and Section4 provides game-theoretic foundations for it. Section 5 characterizes the equilibrium of the OTC model. Section 6 studies the difference between search and bargaining delays. Section 7 provides the asset pricing and liquidity implications of bargaining delays. Section 8 shows how the substitutability of asset classes leads to flights-to-liquidity and adverse effects of gradual transparency policies. Section 9 concludes. All proofs are relegated to the Appendix.

2 Model

Time $t \geq 0$ is continuous. There is a continuum of asset qualities $\theta \in [0, 1]$ each in the fixed unit supply. An asset quality θ brings a flow payoff $d + k\theta$ where $k > 0$ and so, assets of higher quality are associated with higher flow payoffs.[12] The interpretation is that assets are traded within asset classes defined by the public information and k reflects the asset heterogeneity conditional on the public information. Examples of such asset classes are mortgage-backed securities rated AAA maturing in 10 years, investment grade zero-coupon bonds with long maturities issued by biotech firms, renovated studios in downtown area. The quality θ is the index that aggregates various factors that affect asset payoffs, but are not captured by the public information. For example, the MBSs with the same characteristics and credit rating still vary in the default risk determined by the quality of the underlying mortgages, the identity of the bond issuer affects the riskiness of the bond beyond the credit rating and industry, the quality of the residential property is affected by factors not publicly listed such as the safety or expected developments in the area.

There is a continuum of infinitely-lived investors of mass $a > 1$. Investors are risk-neutral and discount at rate r. There are two intrinsic types of investors which we call in anticipation of their equilibrium behavior buyers and sellers. Sellers experience a transitory liquidity shock interpreted as a hedging or liquidity need. For them, holding the asset is associated with additional holding costs which we normalize to 1.[13] Sellers recover from the liquidity shock (and become buyers) with the Poisson intensity y_u, and buyers are hit by the liquidity shock (and become sellers) with the Poisson intensity y_d. Liquidity shocks and recoveries are independent across investors. The initial distribution of intrinsic types is stationary with a mass $\frac{y_u}{y_u+y_d}a$ of buyers and a mass $\frac{y_d}{y_u+y_d}a$ of sellers. To focus on risk-sharing motives, we abstract from investors' portfolio choices and restrict that each investor can hold at most one asset.[14] Assets are initially randomly distributed among investors (Since $a > 1$, not all investors own assets). Investors can borrow and lend freely at interest rate r so that the value of their savings stays bounded below, ruling out Ponzi schemes.

Investors can trade assets in the market with the search friction. Search is costless, and all unmatched investors participate in search. Searching investors are randomly matched to each other. The matching process is independent of the evolution of intrinsic types and is given by the matching technology commonly used in the search-and-bargaining

[12] For real assets, such as real estate, planes, etc., the flow payoff is the instantaneous utility to the asset owner. For financial fixed-income assets, such as corporate bonds, sovereign debt, etc., the flow payoff reflects both the fixed coupon and the risk and costs of default (see Online Appendix for an example).

[13] Flow payoffs and hence prices can be negative, which is interpreted as the buyer is compensated for holding the assets on the balance sheet.

[14] See e.g. Lagos and Rocheteau (2009) for the OTC model with endogenous choice of asset holdings.

literature: Buyers of mass m_b contact sellers of mass m_s with intensity $\frac{\lambda}{2}m_b m_s$, and so the total meeting rate of these two groups of agents is $\lambda m_b m_s$.[15] The contact intensity λ reflects the severity of the search friction.[16]

After the match is found, the investors involved choose whether to participate in the bargaining stage or continue the search. To rule out uninteresting equilibria where the buyer rejects the trade simply because she anticipates that the seller will also reject it, we assume that only buyers are strategic and sellers always choose to participate in the bargaining stage.[17] The buyer can proceed to the bargaining stage with his current match or return to the search stage by saying "yes" or "no", respectively. Note that at this stage no offers are allowed, in particular, we rule out conditional offers, e.g. when the buyer threatens to leave if his offer is not accepted or the seller promises to offer a low prime if the buyer agrees to start the negotiation. This assumption can be motivated by the limited commitment of agents before the bargaining stage starts.

The buyer can condition her strategy on the asset quality θ. The screening bargaining solution describe below is a reduced form for bargaining between investors with private signals about the quality which are infinitely more precise compared to the public information. Thus, the interpretation is that investors condition their strategies on these almost-perfect signals about the asset quality. The mixed strategy of the buyer $\sigma(\theta) \in [0, 1]$ specifies the probability with which the buyer matched with the seller of asset θ participates in the bargaining stage. The interpretation of mixed strategies is that in their decision to proceed to the bargaining stage, investors condition on their idiosyncratic preferences over asset characteristics that do not affect payoffs. After investors proceed to the bargaining stage, they trade an asset θ with delay $t(\theta)$ at price $p(\theta)$. Investors do not participate in search during the bargaining stage,[18] and once agents complete the trade or the intrinsic type of one of the matched agents switches (and there are no gains from trade), the match is destroyed.

Since asset qualities differ only in the level of payoffs, if we set $t(\cdot) \equiv 0$ and let $p(\cdot)$ be determined by the Nash bargaining solution, our model reduces to Duffie et al. (2007). The

[15] Duffie and Sun (2007) provide probabilistic foundations for this matching technology.

[16] The OTC literature also introduces market makers who are able to trade with each other in the frictionless markets and can provide liquidity to the investors. In the recent years, increased capital requirements diminished significantly the role of market makers in markets where they traditionally played an important role, e.g. bonds market, and big banks nowadays prefer to act as brokers matching buyers and sellers to providing the immediacy to their clients.

[17] We will show that in equilibrium the seller always gets a higher utility from bargaining than from continuing the search.

[18] One can think of this restriction as investors cannot make side calls while on the phone negotiating the price with the current match.

key novelty of our approach compared to Duffie et al. (2007) is that instead of using the Nash bargaining solution, we use the novel screening bargaining solution which generally leads to positive, quality-dependent bargaining delays. In the next section, we introduce the screening bargaining solution in a reduced form and microfound it in the subsequent section.

3 Screening Bargaining Solution

We first introduce the screening bargaining solution (SBS) for a general class of bargaining problems and then show how we apply it in our model. Consider the following general bargaining problem described by the tuple (ρ, v, c). For each asset quality $\theta \in [0,1]$, the buyer's valuation is $v(\theta)$ and the seller's cost is $c(\theta)$. In the OTC model, v and c will correspond to endogenous buyer's gains from buying the asset and seller's losses from selling the asset, respectively. Assume that v and c are weakly increasing, almost everywhere continuously differentiable, and the trade surplus $v(\theta) - c(\theta) = \xi > 0$ for all θ.[19] Time is continuous, and parties discount at rate ρ. If parties trade at time t at price p, then the payoff to the buyer is $e^{-\rho t}(v(\theta) - p)$ and the payoff to the seller is $e^{-\rho t}(p - c(\theta))$.

Before formally defining the SBS, let us first provide an intuitive description of how the SBS works out in terms of a related continuous-time bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot)$. The seller continuously decreases her price offer p^s_t starting from p^s_0. The buyer continuously increases his price offer p^b_t starting from p^b_0. Both sides take the paths of offers as given (in particular, it is not conditioned on θ), but choose the time when they accept the offer of the opponent strategically which *is* conditioned on the asset quality θ. The trade happens once one of the sides accepts the price offer of the opponent. Initial offers p^s_0 and p^b_0 has a natural interpretation of bid and ask prices, as by accepting such offers parties can guarantee immediate trade. However, generally parties prefer to wait for a more favorable price offer from the opponent. The continuous-time bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot)$ is a realistic description of the actual negotiations in OTC markets where parties start from extreme price offers and gradually moderate their offers until one of the sides accepts.[20] The next

[19] The assumption that the trade surplus is constant is not necessary for the results in this and the next section, but it simplifies certain steps in the proofs and it holds for the endogenous v and c in the OTC model (see equations $(A.27)$ and $(A.28)$ in Appendix).

[20] E.g. Lewis (2011)(pp. 212-213) describes the negotiation between Morgan Stanley and Deutsche Bank over the price of subprime CDOs:

> What do you mean seventy? Our model says they are worth ninety-five, said one of the Morgan Stanley people on the phone call.
>
> Our model says they are worth seventy, replied one of the Deutsche Bank people.

section shows that $\mathcal{G}(p^s_\cdot, p^b_\cdot)$ is closely related to the standard alternating-offer bargaining game in which parties get very precise signals about the quality.

In the pure-strategy Nash equilibrium of $\mathcal{G}(p^s_\cdot, p^b_\cdot)$, for any asset quality θ corresponds the bargaining outcome consisting of the price $p(\theta)$ and the bargaining delay $t(\theta)$. Of course, the outcome would depend on the choice of paths of price offers $p^s_\cdot$ and $p^b_\cdot$. Let price offers be such that in the equilibrium outcome, the surplus is split proportionally. This uniquely pins down the bargaining delay. We call this equilibrium outcome the SBS defined more formally next.

Definition 1. *The* screening bargaining solution (SBS) (p, t) *to the bargaining problem* (ρ, v, c) *with the surplus split* $\alpha \in (0, 1)$ *satisfies:*

1. *for all* $\theta \in [0, 1]$,
$$p(\theta) = (1 - \alpha)v(\theta) + \alpha c(\theta); \tag{3.1}$$

2. $t(1) = t(0) = 0$ *and for some* θ^**:*
$$\theta \in \operatorname{argmax}_{\theta' \in [\theta^*, 1]} \left\{ e^{-\rho t(\theta')}(v(\theta) - p(\theta')) \right\}, \text{ for } \theta \geq \theta^*, \tag{3.2}$$
$$\theta \in \operatorname{argmax}_{\theta' \in [0, \theta^*]} \left\{ e^{-\rho t(\theta')}(p(\theta') - c(\theta)) \right\}, \text{ for } \theta \leq \theta^*. \tag{3.3}$$

Condition (3.1) states that the price splits the surplus between the buyer and the seller in proportion α to $1 - \alpha$. Thus, the SBS coincides with the Nash bargaining solution in the division of the trade surplus. The difference is the positive trade delay in the SBS characterized by (3.2) and (3.3). For asset qualities above θ^*, in the continuous-time bargaining game the buyer gives in first and accepts the seller's offer at time $t(\theta)$. Condition (3.2) ensures that for the buyer of quality $\theta > \theta^*$ accepting at time $t(\theta)$ is preferred to accepting any other offer corresponding to a different asset quality. Symmetrically, for asset qualities below θ^*, the seller gives in first and accepts the buyer's offer at time $t(\theta)$ and condition (3.3) ensures the optimality of the acceptance time $t(\theta)$ (see Figure 1). In other words, the buyers is screened for high qualities, and the seller is screened for low qualities.

We apply the SBS in the OTC model as follows. We define the *status quo* $(\hat{v}, \hat{c}(\theta))$ as the outcome that gives parties the same utility as they receive during the negotiation

Well, our model says they are worth ninety-five, repeated the Morgan Stanley person, and then went on about how the correlation among the thousands of triple-B-rated bonds in his CDOs was very low, ... he didn't want to take a loss, and insisted that his triple-A CDOs were still worth 95 cents on the dollar.

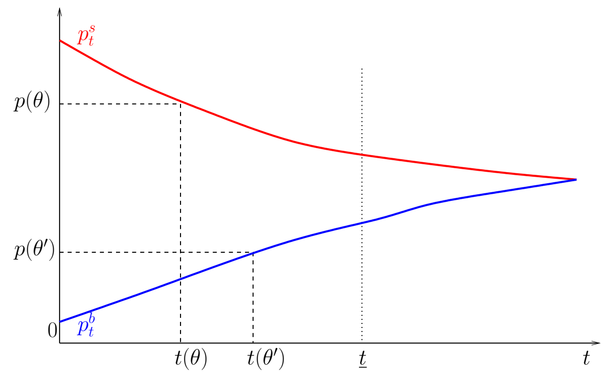


Figure 1: ***Illustration of the SBS.*** *For an asset quality $\theta > \theta^*$, the buyer accepts the seller's offer $p^s_{t(\theta)} = p(\theta)$ at time $t(\theta)$; for an asset quality $\theta' < \theta^*$, the seller accepts the buyer's offer $p^b_{t(\theta')} = p(\theta')$ at time $t(\theta')$.*

process. In Section 5, we introduce value functions $V_{bu}(\theta)$ and $V_{su}(\phi)$ of the unmatched buyer with asset of quality θ and the unmatched seller without asset, resp. Let $v(\theta)$ be the gain from trade for the buyer and $c(\theta)$ be the loss from trade for the seller from the trade, i.e.

$$c(\theta) = \hat{c}(\theta) - V_{su}(\phi), \tag{3.4}$$

$$v(\theta) = V_{bu}(\theta) - \hat{v}. \tag{3.5}$$

Define by $\rho = r + y_u + y_d$ the new discount rate adjusted for the fact that the match can be destroyed if the intrinsic types switch. To determine the price of trade and bargaining delay, we apply the SBS to the bargaining problem (ρ, v, c). We restrict attention to equilibria of the OTC model in which functions v and c are weakly increasing which is a natural assumption given the monotonicity of payoffs. We also assume that functions v and c are continuous on $\Theta_L \cup \Theta_M$, which rules out self-sustained illiquidity. If some asset is expected to trade with a significant bargaining delay, this would to discontinuity in v and c at θ, which could in turn justify these long bargaining delays. While interesting, this mechanism is not the focus of the current paper and we prefer to abstract from it (see e.g. Guerrieri and Shimer (2014) for the related analysis).

4 Microfoundation

This section provides game-theoretic foundations for the screening bargaining solution (SBS).[21]

The generalized Nash bargaining solution (NBS) commonly used in the literature is derived from the static axiomatic approach (Nash (1950), Roth (1979), Binmore (1987)). It predicts the proportional split of the surplus which we refer to as the *Nash split*, but is silent about the delay required to reach this split. Rubinstein (1982) and Binmore et al. (1986) take the non-cooperative approach to show that the split in the NBS is attained without delay when the information about values *is* public and offers are frequent. In this section, we relate the SBS outcome to the outcome of bargaining when the information about values is *almost* public and offers are frequent, and show that in such a model, the trade delay is necessary to attain the Nash split.

Consider the following discrete-time bargaining game $\mathcal{G}(F, \Delta)$. The seller's type θ^s and the buyer's type θ^b are jointly distributed on $[0,1]^2$ according to the CDF F with strictly positive, continuously differentiable density f. Types are affiliated, i.e. f is log-supermodular. We can think of types as noisy private signals about the underlying asset quality θ. The affiliation of signals captures the correlation of signals with the underlying asset quality: a buyer is more likely to receive a high signal θ^b when the asset quality θ is high and thus, the seller's signal θ^s is likely to be high as well. This signal structure is similar to that used in the global games literature (see, e.g., Morris and Shin (1998)). Values are private and given by $v(\theta^b)$ for the buyer and $c(\theta^s)$ for the seller. We assume that functions v and c are strictly increasing, continuously differentiable, and $v(\theta) - c(\theta) = \xi$.[22]

Both sides discount time at a constant rate ρ. Bargaining happens in discrete rounds of length Δ. In the beginning of the round, the seller makes a price offer or accepts the last price offer of the buyer. After delay Δ_b, the buyer either accepts the last price offer of the seller or makes a counter-offer. After that, time $\Delta_s = \Delta - \Delta_b$ elapses and the new round starts. Figure 2 illustrates the bargaining protocol. The ratio $\frac{\Delta_b}{\Delta_b + \Delta_s} = \alpha$ captures the bargaining strength of the buyer.[23] The game stops when one of the parties accepts

[21]Readers more interested in the implications of the bargaining delays for the OTC liquidity may skip it on first reading.

[22]In the equilibrium of our OTC model, functions v and c are only guaranteed to be weakly increasing and they may have discountinuous jumps. Such functions can be approximated (e.g., in L^1 norm) by strictly increasing, continuously differentiable functions, and this section provides microfoundations for this case in the sense that it describes the bargaining outcomes for arbitrarily close specifications of v and c.

[23]This intepretation of α is standard in the bargaining literature (see Osborne and Rubinstein (1990)): the buyer's ability to commit to a longer delay before the counter-offer increases his surplus share in the

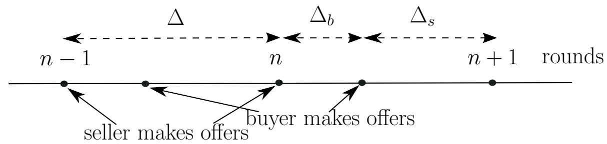


Figure 2: Timing of the discrete-time bargaining game.

the price offer of the opponent with trading happening at the accepted price. Note that as $\Delta \to 0$, parties are able to make offers and respond almost continuously.

The solution concept is Perfect Bayesian equilibrium (PBE). We focus on PBEs in strategies that have the following simple *interval form*: after any history, the set of types that pool with each other and make the same counter-offer (or accept) is an interval. This requirement is stronger than pure strategies, as it rules out strategies in which two types pool with each other, but separate from some types in between. However, it still allows for rich signaling possibilities.[24,25]

We additionally introduce the following refinement. Call a party *informed* after a history if its posterior beliefs assign probability 1 to a single type of opponent. We require that the support of players' posterior beliefs about the opponent's type cannot expand over time, unless there is an informed party, in which case the beliefs of only the informed party are not allowed to expand. The requirement that the support of beliefs does not expand (the support restriction), is standard in the bargaining literature (see Grossman and Perry (1986), Rubinstein (1982), Bikhchandani (1992)). The existing PBE constructions in bargaining games with one-sided uncertainty and two-sided offers, however, do not always satisfy this requirement.[26] In order to guarentee the existence, we slightly weaken the support restriction for the case when one party fully revealed its type.[27]

The SBS is intended to capture bargaining when parties make offers almost continuously and the signals about the quality are very precise. We next formalize this idea. The

complete information game. Similarly, Lemma 4 in Appendix shows that in our model, upper and lower bounds on the price of trade are decreasing in α.

[24]In the bargaining with one-sided private information and two-sided offers (e.g. Gul and Sonnenschein (1988), Ausubel and Deneckere (1989), Grossman and Perry (1986)), the cheap-talk messages that are not accepted, but reveal information are normally ruled out by assumption. The restriction to interval strategies allows for such cheap-talk messages.

[25]As we discuss below in footnote 34 under a stronger notion of the correlation of types, the restriction to interval strategies itself does not prevent achieving the Nash split without delay.

[26]See e.g. Grossman and Perry (1986).

[27]The first part of Theorem 1 below holds without this modification of the support restriction, and we only use it in the proof of the second part (see footnote 39 in the proof of Lemma 12).

(bargaining) outcome consists of a pair of functions (t, p) where $t(\theta^s, \theta^b)$ and $p(\theta^s, \theta^b)$ are the time and price, resp., at which types θ^s and θ^b trade. A PBE outcome is the outcome generated by PBE strategies. For a fixed distribution of types F, consider a sequence of PBE outcomes $(t_{F,\Delta}, p_{F,\Delta})$ indexed by $\Delta \to 0$, and say that $(t_{F,\Delta}, p_{F,\Delta})$ converges to the continuous-time limit (t_F, p_F) if $(t_{F,\Delta}, p_{F,\Delta}) \xrightarrow{p} (t_F, p_F)$.[28,29]

Let F^* be the uniform distribution on the diagonal $\theta^s = \theta^b$. Under F^*, $\theta = \theta^s = \theta^b$ and the quality of the asset is a public information. Consider a sequence of distributions $F \xrightarrow{p} F^*$ such that for any $\varepsilon > 0$,

$$\sup_{(\theta^s,\theta^b):|\theta^s-\theta^b|>\varepsilon} \max\{f(\theta^b|\theta^s), f(\theta^s|\theta^b)\} < \varepsilon$$

for all F sufficiently far in the sequence.[30] When F is close to F^*, the quality is almost public information in the following sense. For any $\varepsilon_0 > 0$, conditional on θ^s, the seller assings probability greater than $1-\varepsilon_0$ to the buyer's type being within ε_0 of θ^s, she assigns a probability greater than $1-\varepsilon_0$ that the buyer assigns probability greater than $1-\varepsilon_0$ that the seller's type is within ε_0 of θ^s and we can continue these statements up to perhaps very large (for F very close to F^*), but necessarily finite order (hence, the information about the quality is almost public, but not public).

The main result of this section is that the bargaining outcomes are quite different when θ is public information ($F = F^*$), and when it is almost public information ($F \approx F^*$). Let us start with the former. Denote the price of the Nash split by $\mathbf{p}(\theta^s, \theta^b) = (1-\alpha)v(\theta^b) + \alpha c(\theta^s)$. The following proposition due to Binmore et al. (1986) states that in this case trade happens without delay.

Proposition 1 (Binmore et al. (1986)). *(t_{F^*}, p_{F^*}) does not depend on the sequence $(t_{F^*,\Delta}, p_{F^*,\Delta})$ and $(t_{F^*}, p_{F^*}) = (0, \mathbf{p})$.*

Consider now the case of almost public information about the asset quality. Denote by $\mathbf{t}(\theta^s)$ the delay associated with quality θ^s in the SBS as given by (3.2) – (3.3).

Theorem 1. *1. Consider a sequence of PBE continuous-time limits (t_F, p_F) indexed by $F \xrightarrow{p} F^*$. If $p_F \xrightarrow{p} \mathbf{p}$ as $F \xrightarrow{p} F^*$, then there exist $0 < x_l < x_h < 1$ and*

[28]Here and further, $\xrightarrow{p}$ denotes convergence in probability, e.g. $(t_{F,\Delta}, p_{F,\Delta}) \xrightarrow{p} (t_F, p_F)$ as $\Delta \to 0$ if for all $\varepsilon > 0$, $\lim_{\Delta\to 0} \mathbb{P}_F (|t_{F,\Delta} - t_F| < \varepsilon$ and $|p_{F,\Delta} - p_F| < \varepsilon) = 1$.

[29]There are known technical issues in definining games in continuous time (see Simon and Stinchcombe (1989)). For this reason, it is standard in the bargaining literature to take a limit $\Delta \to 0$ in the discrete-time game to obtain predictions that do not depend on the protocol of bargaining (e.g. the order of offers).

[30]See Online Appendix for an example of such a sequence.

$0 < \underline{\theta}^s < \overline{\theta}^s < 1$ *such that*

$$x_l > \limsup_{F \xrightarrow{p} F^*} \mathbb{E}_F[e^{-\rho t_F}], \tag{4.1}$$

$$x_h > \liminf_{F \xrightarrow{p} F^*} \mathbb{E}_F[e^{-\rho t_F} | \theta^s < \underline{\theta}^s \text{ or } \theta^s > \overline{\theta}^s]. \tag{4.2}$$

2. *Suppose* $c(1) - c(0) < \min\{\frac{1-\alpha}{\alpha}, \frac{\alpha}{1-\alpha}\}\xi$. *Then there exists a sequence of PBE continuous-time limits* (t_F, p_F) *such that* $(t_F, p_F) \xrightarrow{p} (\mathbf{t}, \mathbf{p})$ *as* $F \xrightarrow{p} F^*$.[31]

Theorem 1 shows that the bargaining outcome when information the asset quality is not public differs drastically from the case when the quality is a pubic information, and importantly, this difference does not vanish as players's signals become very precise (F arbitrarily close to F^*). First, the bargaining delay is necessary to attain the Nash split of the surplus (inequality (4.1)). Second, the bargaining delay is generally non-monotone: it is lower for qualities closer to extremes of the distribution (0 and 1) and higher in the middle (inequality (4.2)). Finally, when the surplus from trade is sufficiently large, the SBS can be approximated by the PBE outcomes as offers become frequent and the information about quality becomes almost public. As we will show in the next section, the SBS outcome captures positive and non-monotonic bargaining delays (U-shaped liquidity) that we obtain in Theorem 1.

Let us provide the intuition for these results. First, why the delay is necessary to attain the Nash split? Although the formal proof is quite involved, the underlying idea is simple. Suppose to contradiction, for any Δ and F arbitrarily close to 0 and F^*, resp., there were a PBE in which trade happens with high probability without a significant delay. Since sufficiently different asset qualities should be traded at sufficiently different prices in order to match the Nash split, it is necessary that at least one of the sides reveals quite precisely its signal. Then one side, say the buyer, can relatively quickly convince the seller that its value is relatively low. But this implies that high types of the buyer can mimic lower types and get a more favorable price by only slightly delaying the trade, which is a contradiction to the sequential rationality.[32]

The non-monotonicity of the bargaining delay is also quite intuitive. We show that the buyer has the option to trade immediately at price close to $\mathbf{p}(1)$ which is the complete-

[31] That is, for all $\varepsilon > 0$,$\lim_{F \xrightarrow{p} F^*} \mathbb{P}_F\left(|t_F - \mathbf{t}| < \varepsilon \text{ and } |p_F - \mathbf{p}| < \varepsilon\right) = 1$.

[32] Here, the assumption that the support cannot expand is crucial: once the buyer signals that his value is relatively low, this gives him a guarantee of relatively low price in any continuation play. Lemma 4 in Appendix shows how the bounds on the price of trade depend on the support of types remaining in the game.

information price of trade when both players' types equal 1.[33] Since $p_F \xrightarrow{p} \mathbf{p}$, the buyer's types close to 1 expect to trade with a high probability at a price close to $\mathbf{p}(1)$, thus, for them the expected bargaining delay cannot be too long. Symmetric argument shows that for the seller's types close to 0, the expected bargaining delay is relatively show, as they have the option to trade immediately at a price close to $\mathbf{p}(0)$ (which is the complete-information price of trade when both players' types equal 0). Therefore, types close to the extremes of the range are guaranteed to trade relatively quickly which gives us inequality (4.2).

Now, let us turn to why the SBS can be approximated by the PBEs for $F \approx F^*$, but not when $F = F^*$? When the information about the quality is public, there is a unique split of the surplus sustainable in any continuation equilibrium and so, it is not possible to reward or punish players to sustain the delay. This is, however, possible when the information is noisy. We construct PBEs approximating the SBS in the grim trigger strategies. In particular, we specify that if e.g. the seller deviates from the equilibrium path, then the buyer infers that the seller's signal is very low and the seller is very desperate to trade (formally, the buyer beliefs that the seller's type is 0). After such an optimistic updating, in the continuation equilibrium the buyer almost immediately gets the maximal share of the surplus. By specifying such a punishment path, we can sustain the equilibrium path that involves delay. Despite the fact there is an efficiency loss due to the bargaining delay on the equilibrium path and both parties assign a high probability to it, nobody wants to seem desperate and deviate from the equilibrium path.

Theorem 1 highlight the crude public information, rather than parties' private information, as the source of the bargaining delay. The assumption of the coarse public information about the asset quality is relevant in many OTC markets. Credit ratings for financial assets put only crude bounds on the risks associated with the asset, and experienced traders rely on their private information sources to further refine these bounds. Likewise, in the real estate, an experienced realtor goes beyond the public profile of the house and assesses various characteristics of the neighborhood, such as safety and demographics, to determine more precisely its value. In our model, the precision of the public information is captured by the slope of functions v and c (or referring to the primitives of the OTC model by the parameter k): the more homogenous the assets, the smaller the differences in the prices at which assets trade, and in turn, the smaller bargaining delay is required to attain the Nash split.

Let us motivate our focus on the PBEs of the bargaining game approximating the

[33] SeeLemma 4 in Appendix.

SBS. The multiplicity of equilibria is the major concern in the literature on bargaining with two-sided private information about values and two-sided offers. In particular, in our model, along with PBEs described in Theorem 1, there is a continuum of PBEs in which trade happens immediately at some price in $(c(1), v(0))$. We choose to focus on PBEs that approximate the Nash split. On the one hand, this allows us to better contrast our results to that in the OTC literature which applies the Nash bargaining solution. On the other hand, the original paper by Nash (1950) gives axiomatic foundations for the Nash split of the surplus. Thus, it is reasonable to assume that such a split would be a natural focal point for equilibrium split when players have very precise information about the quality.[34]

Technically, the sharp constast between the outcomes with public and almost public information about the asset quality stems from the order of limits. Rubinstein (1982)'s analysis first assumes that the idealized complete-information model $\mathcal{G}(F^*, \Delta)$ is a good approximation for $\mathcal{G}(F, \Delta)$, and then makes offers frequent. Our analysis first makes offers frequent in $\mathcal{G}(F, \Delta)$, and then takes limit $F \xrightarrow{p} F^*$. We beliefve that our approach better captures the negotiation in OTC markets where there are virtually no restrictions on the protocol of bargaining, while the private information of parties is a relevant feature. Relatedly, the proof of Theorem 1 suggests that the limiting case ($F \to F^*$) sheds light also on the bargaining dynamics when the noise in the signals is not small. We consider the continuous-time bargaining game introduced in the previous section for any distribution F (not only for F^* as in the SBS) and show that the trade dynamics similar to that in the SBS arises in BNEs of this game.

5 Equilibrium

This section characterizes the equilibrium of the OTC model.

[34] Tsoy (2014) shows that the SBS outcome is robust to the assumptions about the distribution of types. Tsoy (2014) considers F with positive mass only on a band around the diagonal $\theta^s = \theta^b$ and approximate F^* by making this band very narrow. This correlation structure is much stronger that the one used in this paper in that players know, rather than assign very high probability, that their signals are close to each other. Under this stronger correlation structure, Tsoy (2014) constructs PBEs approximating the SBS.

Under this stronger correlation structure and dispensing with the assumption that the support of beliefs does not expand, Tsoy (2014) also contructs PBEs that attain the Nash split without delay. On the one hand, Theorem 1 shows that such an outcome is not robust to the model of the correlation used in this paper. On the other hand, the construction of the efficient outcome in Tsoy (2014) also uses interval strategies suggesting that such strategies are not very restrictive.

Steady-State Equilibrium We first describe the distribution of asset holdings among investors and define the steady-state equilibrium. At every time, each investor can be either matched (m) or unmatched (u). We refer to the intrinsic type of the investor and his match status as the *type* $\tau \in \{bu, su, bm, sm\}$. The *asset position* of the investor $[0,1] \cup \{\phi\}$ is the quality of the asset that the investor owns or bargains over (ϕ denotes investors who neither own nor bargain over an asset). The evolution of types and asset holdings is depicted in Figure 3. The distribution of assets among different types of investors at time t is given by $M_t = \{M_{\tau,t} \in \Delta([0,1]), \tau \in \{bm, bu, sm, su\}\}$ where for any measurable set $\Theta \subseteq [0,1]$, $M_{\tau,t}(\Theta)$ gives the mass of type τ investors with asset positions in Θ. We suppose that there is the mass density function $\mu_{\tau,t}$ of $M_{\tau,t}$ such that $M_{\tau,t}(\Theta) = \int_\Theta \mu_{\tau,t}(\theta)d\theta$. M_t satisfies following balance conditions:

1. for any θ, the mass of investors with the asset position θ is equal to the (unit) supply of the asset of quality θ,

$$\mu_{su,t}(\theta) + \mu_{bu,t}(\theta) + \mu_{bm,t}(\theta) = 1; \tag{5.1}$$

2. the mass of investors with the asset position ϕ is equal to $a - 1$,

$$M_{su,t}(\phi) + M_{bu,t}(\phi) + M_{bm,t}([0,1]) = a - 1; \tag{5.2}$$

3. for any θ, the masses of matched buyers and matched sellers coincide,

$$\mu_{sm,t}(\theta) = \mu_{bm,t}(\theta). \tag{5.3}$$

We focus on steady states of the economy in which $\frac{\partial}{\partial t} M_{\tau,t}(\Theta) = 0$ for all Θ and τ, and in what follows, we omit from the notation the subscript t. The definition of the (steady-state) equilibrium of the OTC model is as follows.

Definition 2. *A tuple $(\sigma(\cdot), M)$ constitutes an* equilibrium *if the buyer's strategy $\sigma(\cdot)$ is optimal given M, and M is the stead-state distribution of assets generated by $\sigma(\cdot)$.*

Equilibrium Characterization We now characterize the equilibrium of the OTC model. We first introduce several quantities central in the characterization. Denote $\Theta_L = \{\theta : \sigma(\theta) = 1\}$, $\Theta_I = \{\theta : \sigma(\theta) = 0\}$, and $\Theta_M = \{\theta : \sigma(\theta) \in (0,1)\}$. Call assets in Θ_L *unconditionally liquid* or simply *liquid*, assets in Θ_M *conditionally liquid*, and assets in

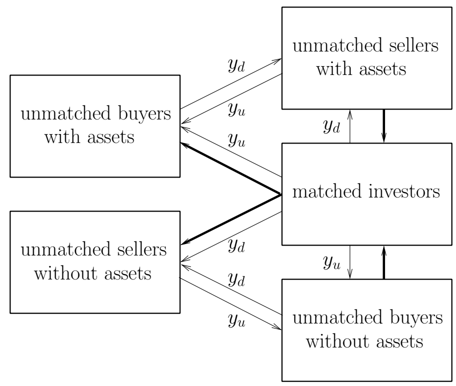


Figure 3: ***The evolution of types and asset holdings.*** *Bold arrows indicate transitions between types and changes in asset holding caused by the start or end of bargaining, and thin arrows indicate transitions caused by the switching of the intrinsic types (intensities are written next to arrows). For example, a matched seller of the asset with quality θ can either recover from the liquidity shock, or lose her match because the buyer is hit by a liquidity shock, or complete the trade after the bargaining delay and become an unmatched seller.*

Θ_I *illiquid.* Let $\Lambda_s = \lambda M_{bu}(\phi)$ be the Poisson intensity of contact with unmatched buyers without assets, and $\Lambda_b = \lambda M_{su}(\Theta_L)$ be the Poisson intensity of contact with sellers of liquid assets. Both are measures of market thickness and capture how easily each side of the market can find a trade partner. We will show that two are closely related, and by convention, we will only refer to Λ_s as the market thickness. Let $F_L \in \Delta(\Theta_L)$ be the steady-state probability distribution of qualities in the pool of unmatched sellers of liquid assets and $L = |\Theta_L|$ be the mass of liquid assets. Let $x(\theta) = e^{-\rho t(\theta)}$ be the factor by which the surplus from trade of the asset θ is dissipated due to the bargaining delay. We refer to $x(\theta)$ as the liquidity of asset θ.

Let us outline the steps of the analysis. First, In Lemma 1 we derive for fixed $x(\cdot)$ and $\sigma(\cdot)$ the steady-state distribtuion M. Then for fixed $x(\cdot)$ and M we derive investors' value functions and optimal strategy $\sigma(\cdot)$. Finally, in Lemma 3, given investors' value functions we derive the liquidity profile $x(\cdot)$. Generally, finding the equilibrium would require solving a system of functional equations in $(M, \sigma(\cdot), x(\cdot))$. However, we show that the equilibrium analysis can be reduced to finding the market thickness Λ_s and the mass of liquid assets L which pin down M and $\sigma(\cdot)$.

Step 1: We first characterize the unique steady-state distribution M.

Lemma 1. *For any $\sigma(\cdot)$ and $x(\cdot)$, there exists a unique steady-state distribution M characterized by $(\sigma(\cdot), x(\cdot), \Lambda_s)$, and*

- *F_L is uniform on Θ_L;*

- *Λ_s is the unique solution to*

$$\frac{\Lambda_s}{\lambda} = \frac{y_u(a-1)}{y_u+y_d} - \frac{y_d}{y_u+y_d}\int_0^1 \frac{\Lambda_s\sigma(\theta)}{y_u+y_d+\Lambda_s\sigma(\theta)}d\theta; \tag{5.4}$$

- *Λ_b is given by*

$$\Lambda_b = \frac{\lambda y_d L}{y_u+y_d+\Lambda_s}. \tag{5.5}$$

Moreover, a pointwise weak increase in $\sigma(\cdot)$ leads to a decrease in Λ_s and and increase in Λ_b.

The steady-state distribution of assets is pinned down by the market thickness Λ_s, and moreover, Λ_s, Λ_b, and F_L depend only on $\sigma(\cdot)$, but not on $x(\cdot)$.[35] The fact that F_L coincides with the distribution of assets' supply conditional on $\theta \in \Theta_L$ greatly simplifies equilibrium analysis and may be surprising at first sight. Indeed, one may expect that assets with shorter bargaining delays are more abundant in the market and F_L should reflect this. To see why it is not so, note that the inflow into the group of sellers of asset of quality θ is formed from matched sellers of an asset of quality θ whose counter-party is hit by a liquidity shock and from unmatched buyers owning an asset of quality θ who are hit by a liquidity shock (see Figure 3). Both these inflows have intensity y_d. At the same time, the outflow from this group of sellers happens because of the recovery from the shock of sellers and the formation of new matches. The former has intensity y_u and the latter has intensity Λ_s, and both are again independent of bargaining delays $t(\cdot)$. Therefore, $x(\cdot)$ only affects the distribution of investors between those who have already completed the trade and those still bargaining, but does not affect the distribution of searching sellers of liquid assets.

To interpret equation (5.4) suppose $|\Theta_M| \approx 0$. Then

$$M_{bu}(\phi) \approx \frac{y_u(a-1)}{y_u+y_d} - \lambda L \frac{y_d}{y_u+y_d} M_{bu}(\phi)\frac{1}{y_u+y_d+\Lambda_s}, \tag{5.6}$$

that is, the mass of unmatched buyers without assets ($M_{bu}(\phi)$) equals the total mass of buyers without assets ($\frac{y_u(a-1)}{y_u+y_d}$) minus the term proportional to the number of matches between sellers of liquid assets (of mass $L\frac{y_d}{y_u+y_d}$) and unmatched buyers without assets (of mass $M_{bu}(\phi)$) (the adjustment term accounts for the fact that some of the sellers

[35]Unlike Λ_s,Λ_b,F_L, the steady-state distribution M derived explicitly in the Appendix depends on the profile $x(\cdot)$.

are already matched). Notice that the fact that buyers may reject some asset qualities in equilibrium results in the endogenous length of search delays. The increase in $\sigma(\cdot)$ results in an increase in the likelihood of finding a match for buyers and a decrease in the likelihood of finding a match for sellers. In equation (5.6), the search intensity is adjusted and effectively is λL adjusted for the fact that only a fraction L of assets is accepted for trade by buyers.

Note that Lemma 1 holds for any $\sigma(\cdot)$ and $x(\cdot)$, not necessarily those analysed in this paper, and can be used to study other forms of post-match trade delay, such as the time for parties to familiarize themselves with the asset, time to execute the trade, etc.

Step 2: Now, given the steady-state distribution M and $x(\cdot)$, we determine investors' value functions (for each type τ and asset position) and equilibrium $\sigma(\cdot)$. For $\tau \in \{bu, su, bm, sm\}$, let $V_\tau(\theta)$ be the expected utility of an investor of type τ owning (or bargaining over) asset θ, and for $\tau \in \{bu, su\}$, let $V_\tau(\phi)$ be the expected utility of an investor of type τ owning no asset. Value functions during the search stage are determined by the following Bellman equations,

$$\begin{aligned}
rV_{su}(\phi) &= y_u(V_{bu}(\phi) - V_{su}(\phi)), &(5.7)\\
rV_{bu}(\theta) &= d + k\theta + y_d(V_{su}(\theta) - V_{bu}(\theta)), &(5.8)\\
rV_{bu}(\phi) &= y_d(V_{su}(\phi) - V_{bu}(\phi)) + \Lambda_b\left(\mathbb{E}\left[V_{bm}(\theta)|\theta \in \Theta_L\right] - V_{bu}(\phi)\right), &(5.9)\\
rV_{su}(\theta) &= d + k\theta - 1 + y_u(V_{bu}(\theta) - V_{su}(\theta)) + \sigma(\theta)\Lambda_s\left(V_{sm}(\theta) - V_{su}(\theta)\right). &(5.10)
\end{aligned}$$

The depreciation of value functions in the left-hand side of equations (5.7) – (5.10) equals the sum of flow payoffs and changes in value functions due either to switches of intrinsic types or the formation of matches. For example, in equation (5.9), the flow payoff of the searching buyer without an asset is zero. If the buyer is hit by a liquidity shock, his value function drops to $V_{su}(\phi)$, while if he is matched to a seller, then his value function increases to $\mathbb{E}\left[V_{bm}(\theta)|\theta \in \Theta_L\right] = \int_{\Theta_L} V_{bm}(\theta)dF_L(\theta)$. Notice that if a buyer is matched to a seller of an asset in Θ_M, then his continuation utility is $V_{bu}(\phi)$ irrespective of whether he starts to negotiate or continues to search. Therefore, in equation (5.9), it is sufficient to consider only assets in Θ_L. This is however not the case in equation (5.10) which described the value function of the unmatched seller of asset quality θ. In equilibrium, such a seller strictly prefers to start the negotiation, and hence, for her the probability with which the asset that she offers for trade is accepted is important.

To determine $V_{bm}(\theta)$ and $V_{sm}(\cdot)$, let $V_{bm}(t, \theta)$ and $V_{sm}(t, \theta)$ be the value functions of the matched seller and buyer, resp., when time t has passed since the start of bargaining.

Functions $V_{bm}(t,\theta)$ and $V_{sm}(t,\theta)$ satisfy the Bellman equations

$$rV_{bm}(t,\theta) = y_u(V_{bu}(\phi) - V_{bm}(t,\theta)) + y_d(V_{su}(\phi) - V_{bm}(t,\theta)) + \tfrac{\partial}{\partial t}V_{bm}(t,\theta), \tag{5.11}$$

$$rV_{sm}(t,\theta) = d + k\theta - 1 + y_u(V_{bu}(\theta) - V_{sm}(t,\theta)) + y_d(V_{su}(\theta) - V_{sm}(t,\theta)) + \tfrac{\partial}{\partial t}V_{sm}(t,\theta). \tag{5.12}$$

Equation (5.11) reflects that with intensity y_u and y_d the intrinsic type of one side switches and the value function becomes $V_{bu}(\phi)$ and $V_{su}(\phi)$, resp., and the value function increases over time as the time of trade $t(\theta)$ approaches (equation (5.12) is analogous). After parties have spent time $t(\theta)$ in the bargaining stage, they trade and get utilities $V_{bm}(t(\theta),\theta) = V_{bu}(\theta) - p(\theta)$ and $V_{sm}(t(\theta),\theta) = p(\theta) + V_{su}(\phi)$, which are the terminal conditions for (5.11) and (5.12), resp. With $c(\theta)$ and $v(\theta)$ and defined in (3.4) and (3.5) we can use the proportional split of the surplus in the SBS to determine $p(\theta)$. In the Appendix, we show that the trade surplus $v(\theta) - c(\theta)$ is equal to $\xi \equiv \frac{1}{\rho}$ for all θ which follows from the fact that holding costs do not depend on the asset quality.

We can combine Bellman equations (5.7)−(5.10) and (5.11)−(5.12) and the expression for the price $p(\theta)$ to express value functions through $x(\theta)$ and $\sigma(\theta)$. This allows us to determine optimal strategies described in the following lemma.

Lemma 2. *The asset of quality θ is always accepted by buyers ($\theta \in \Theta_L$), whenever*

$$x(\theta) > \underline{x} \equiv \frac{\Lambda_b}{\rho + \Lambda_b}\left(\frac{1}{L}\int_{\Theta_L} x(\theta)d\theta\right), \tag{5.13}$$

and it is always rejected ($\theta \in \Theta_I$) whenever the inequality in (5.13) *is reversed. Moreover,*

$$V_{bu}(\phi) = \alpha\frac{r + y_u}{r}\xi\underline{x}. \tag{5.14}$$

From Lemma 2, buyers search for sufficiently liquid assets in the market. If not all assets are accepted for trade in equilibrium, there is a non-trivial search process occurring in equilibrium. The buyer may reject several assets before he finds a sufficiently liquid asset for which he proceeds to the bargaining stage. The threshold of the buyer depends on the average liquidity and the ability to find liquid assets in the market. If the search is fast (Λ_b is large), then the buyer's threshold is close to the average liquidity, i.e. the outside option of the buyer is essentially to go back to the market and get a random draw from the pool Θ_L. If the search is slow (Λ_b is small), then the buyer accepts a wide range of assets, as finding another asset entails a significant delay.

Note that buyers trade-off search and bargaining delays and by varying their strategy $\sigma(\cdot)$ they effectively control the length of their search delays. When buyers accept a smaller range of assets for trade, their search delays increase as they reject more assets and it takes longer time to find "sufficiently" liquid asset.

Step 3: Finally, we use the definition of the SBS to determine the liquidity profile $x(\cdot)$ for given M and $\sigma(\cdot)$.

Lemma 3. *Either $\Theta_L = [0,1]$ or there exist $0 < \check{\theta} < \underline{\theta} \leq \theta^* \leq \hat{\theta} < 1$ such that $\Theta_L = [0,\check{\theta}] \cup [\hat{\theta},1]$ and $\Theta_M = (\check{\theta},\underline{\theta})$. Moreover,*

$$x(\theta) = \begin{cases} \exp\left(\dfrac{v(1)-v(\theta)}{\alpha\xi}\right), & \text{for } \theta > \hat{\theta}, \\ \exp\left(-\dfrac{c(\theta)-c(0)}{(1-\alpha)\xi}\right), & \text{for } \theta \leq \check{\theta}. \end{cases} \tag{5.15}$$

The next theorem combines equilibrium conditions for M, σ, x derived above to characterize the equilibrium and provide conditions for existence and uniqueness.

Theorem 2. *The equilibrium of the OTC model is characterized by the unique solution (Λ_s, L) to*

$$\begin{cases} \Lambda_s \geq \dfrac{\lambda y_d}{\rho}\left(\dfrac{\xi r}{k}\left(e^{\frac{k}{\xi r}L}-1\right)-L\right)-y_u-y_d, & \text{with equality iff } L<1 \quad (5.16) \\ \dfrac{\Lambda_s}{\lambda} = \dfrac{y_u(a-1)}{y_u+y_d} - \dfrac{y_d}{y_u+y_d} I(L,\Lambda_s), & \quad (5.17) \end{cases}$$

where

$$I(L,\Lambda_s) = \frac{\Lambda_s L}{y_u+y_d+\Lambda_s} + (1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}e^{-\frac{k}{r\xi}L}\int_0^{\min\left\{\frac{\rho+\Lambda_s}{\Lambda_s}\frac{(1-L)k}{(1-\alpha)y_d\xi}e^{\frac{k}{r\xi}L},1\right\}} \frac{1-s}{1+\frac{y_u+y_d}{\Lambda_s}-\frac{r}{\rho}s}ds. \tag{5.18}$$

The equilibrium exists whenever $\alpha \geq \frac{y_d}{r+y_d}$ and is unique if in addition $1-\alpha < \frac{r^2}{y_d(r+y_u+y_d)}$. Moreover, the equilibrium distribution of asset holdings M, liquidity profile $x(\cdot)$ and strategy $\sigma(\cdot)$ do not depend on the level of flow payoffs d.

The equilibrium is pinned down by the market liquidity L and the market thickness Λ_s. Equation (5.16) describes the optimal buyers' strategy (derived in Lemma 2) and it gives an increasing relationship between Λ_s and L. When the sellers' contact intensity is higher, it is harder for buyers to find trade partners (see equation (5.5)) and so, they optimally

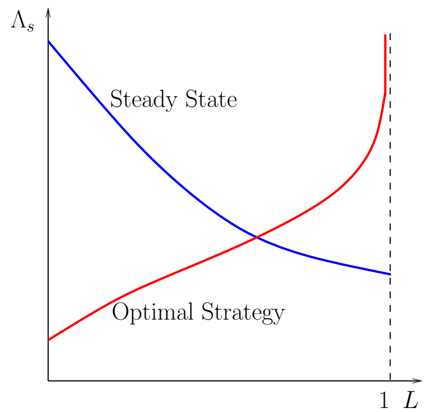


Figure 4: Equilibrium is determined as the intersection of the increasing curve given by equation (5.16) *reflecting the optimality of strategy* $\sigma(\cdot)$ *and the decreasing curve given by equation* (5.17) *reflecting the stead-state distribution of assets in the economy.*

expand the range of acceptable assets. Equation (5.17) describes the steady-state market thickness (derived in Lemma 1) and it gives a decreasing relationship between Λ_s and L. When buyers accept more assets for trade, fewer buyers are searching in the market, as more of them have already traded or are in the process of negotiation. Thus, it is harder for the sellers to find a trade partner and Λ_s decreases. In the Appendix, we show when the seller's share is not too high ($1-\alpha < \frac{r^2}{y_d(r+y_u+y_d)}$), both relations are strictly monotone proving the uniqueness of the equilibrium (see Figure 4).

Observe that the level of flow payoffs does not affect the liquidity. Thus, in our liquidity analysis, we interpret k as capturing the heterogeneity of assets conditional on the public information and abstract from the fact that it also changes the level of asset payoffs. In particular, an increase in k can be accompanied by a decrease in d so that the average level of asset payoffs is the same (or even drops) and this would not affect our conclusions about the effect of k on liqudity.

6 Are Bargaining and Search Delays Different?

In this section, we study the difference between the search and bargaining delays by looking at the limits when only one type of delays is present. In the following proposition, we mark with one star equilibrium quantities in the limit $k \to 0$ and with two stars equilibrium quantities in the limit $\lambda \to \infty$.

Proposition 2. *1. The limit of the equilibrium as* $k \to 0$ *is characterized by* $L^\star =$

$1, \underline{x}^\star = \overline{x}^\star = 1$, *and* $\Lambda_s^\star$ *solving:*

$$\frac{\Lambda_s^\star}{\lambda} = \frac{y_u(a-1)}{y_u+y_d} - \frac{y_d}{y_u+y_d}\frac{\Lambda_s^\star}{y_u+y_d+\Lambda_s^\star}. \tag{6.1}$$

Moreover, $\lim_{\lambda\to\infty}\Lambda_s^\star = \infty$ *only if* $\frac{y_u}{y_u+y_d}a > 1$, *and* $\lim_{\lambda\to\infty}\Lambda_s^\star < \infty$ *only if* $\frac{y_u}{y_u+y_d}a < 1$.

2. *The limit of the equilibrium as* $\lambda \to \infty$ *is characterized by* $(L^{\star\star}, \Lambda_s^{\star\star})$ *satisfying:*

   (a) *If* $L^{\star\star} \in (0,1)$, *then* $\underline{x}^{\star\star} < 1, \overline{x}^{\star\star} < 1, \Lambda_b \to \Lambda_b^{\star\star} < \infty, \Lambda_s \to \infty$ *and*

$$\lim_{\lambda\to\infty}\frac{\Lambda_s}{\lambda} \geq \frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L^{\star\star}}-1\right) - L^{\star\star}\right), \text{ with equality iff } L^{\star\star} < 1, \tag{6.2}$$

$$\lim_{\lambda\to\infty}\frac{\Lambda_s}{\lambda} = \frac{y_u(a-1)}{y_u+y_d} - \frac{y_d}{y_u+y_d}L^{\star\star} - \frac{(1-\alpha)y_d^2\xi}{(y_u+y_d)k}e^{-\frac{k}{r\xi}L^{\star\star}}\int_0^{\min\left\{\frac{(1-L^{\star\star})k}{(1-\alpha)y_d\xi}e^{\frac{k}{r\xi}L^{\star\star}},1\right\}}\frac{\rho(1-s)ds}{\rho-rs}. \tag{6.3}$$

   Moreover, $L^{\star\star} = 1$ *iff* $\frac{y_u}{y_u+y_d}a \geq 1 + \frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}}-1\right)-1\right)$.

   (b) *If* $L^{\star\star} = 0$, *then* $\underline{x}^{\star\star} = \overline{x}^{\star\star} = 1, \Lambda_s^{\star\star} < \infty, \Lambda_b \to \infty$ *and*

$$y_u(a-1) = \frac{(1-\alpha)y_d^2\xi}{k}\frac{\Lambda_s^{\star\star}}{\rho+\Lambda_s^{\star\star}}\int_0^{\min\left\{\frac{\rho+\Lambda_s^{\star\star}}{\Lambda_s^{\star\star}}\frac{k}{(1-\alpha)y_d\xi},1\right\}}\frac{\Lambda_s^{\star\star}(1-s)}{y_u+y_d+\Lambda_s^{\star\star}\left(1-\frac{r}{\rho}s\right)}ds. \tag{6.4}$$

   Moreover, $|\Theta_I| > 1 - \frac{y_d\xi}{k}$ *and* $L^{\star\star} = 0$ *iff* $\frac{y_u}{y_u+y_d}a < 1$.

   (c) *If* $\Theta_I \neq \emptyset$ *for some* λ, *then* $L^{\star\star} < 1$.

When there is no bargaining friction and assets are homogeneous ($k = 0$), the model reduces to Duffie et al. (2007). Since assets are identical, there is no bargaining delay and all assets are accepted for trade. In this case, search delays are random and do not vary across assets. For $k = 0$, as $\lambda \to \infty$ all potential trades are executed without delay. E.g. when $\frac{y_u}{y_u+y_d}a > 1$ and so buyers are relatively abundant, sellers are able to immediately liquidate their positions. When $\frac{y_u}{y_u+y_d}a < 1$ and so sellers are relatively abundant, some sellers are rationed and have to hold their positions.

When the bargaining friction is present and assets are heterogeneous ($k = 0$), even when the search friction vanishes, the outcome need not be efficient. Differences in endogenous bargaining delays translate into differences in liquidity $x(\cdot)$ which persist even as the search friction vanishes. Because of this even when buyers are relatively abundant ($\frac{y_u}{y_u+y_d}a > 1$), the bargaining takes time ($\overline{x}^{\star\star} < 1$) and trades are not executed immediately. The buyers are willing to tolerate bargaining delays because there is an endogenous shortage of sellers, so buyers spend in expectation $\frac{1}{\Lambda_b^{\star\star}}$ to get to trade with the seller of a liquid asset. Interestingly, despite the congestion on the buyer's side, buyers may reject some assets when $\frac{y_u}{y_u+y_d}a < 1 + \frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}} - 1\right) - 1\right)$. For rejected assets, buyers prefer to search further for more quickly negotiated assets. When $\frac{y_u}{y_u+y_d}a < 1$ and buyers are relatively rare, sellers are relatively abundant and buyers can quickly search the market for the most liquid asset ($\Lambda_b^{\star\star} = \infty$). Thus, buyers accept only assets associated with virtually no bargaining delay ($\underline{x}^{\star\star} = 1$). When k is large ($k > y_d\xi$), there is a range of assets that buyers reject in their search.

Proposition 2 shows that bargaining delays vary from search delays in several respects. First, bargaining delays create an endogenous heterogeneity in liquidity and lead to the extensive trade margin, not present in the random-search delay models. Second, the extensive trade margin leads to an endogenous length of search for buyers. In their search process, they can reject several assets until they find a sufficiently liquid asset. Third, on the intensive margin, bargaining delays are deterministic in our model conditional on the quality of the asset.

Let us mention that the extensive trade margin can be interpreted both literally as assets in Θ_I are not traded in the market and as assets in Θ_I are more sensitive to market conditions, e.g. the selling pressure. This interpretation is formalized in the following proposition.

Proposition 3. *Suppose $\Theta_I \neq \emptyset$. Consider an increase in y_d and a decrease in y_u so that $y_u + y_d$ stays constant. Then L decreases.*

In Proposition 3, the selling pressure is the increase in the proportion of sellers in the market captured via a simultaneous offsetting increase in y_d and decrease in y_u, so that the long-run ratio of sellers in the population increases. The selling pressure results in a wider range of assets being illiquid. Thus, assets in Θ_I may be traded during normal times, but the ability to sell such assets is impaired when market conditions worsen, e.g. occurrence of the selling pressure.

7 Bargaining Delays and Liquidity

This section derives implications of bargaining delays for asset liquidity and prices.

Liquidity Measures We focus on the following liquidity measures. First, the screening bargaining solution is the equilibrium outcome of the bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot)$. In $\mathcal{G}(p^s_\cdot, p^b_\cdot)$, each side can immediately accept the first offer of the opponent and so, first offers p^b_0 and p^s_0 are essentially the bid and ask prices, resp., and their difference $ba = p^s_0 - p^b_0$ is the bid-ask spread.

Second, we look at how liquidity varies across assets. The extensive margin is captured by $\sigma(\theta)$, the probability that the buyer accepts the asset. The intensive margin is captured by the cost of bargaining delay $x(\theta) = e^{-\rho t(\theta)}$ (cost is higher when $x(\theta)$ is smaller) and the (instantaneous) trade volume which equals[36]

$$w(\theta) = \frac{\Lambda_s \sigma(\theta) y_d}{y_u + y_d + \Lambda_s \sigma(\theta)} x(\theta)^{\frac{y_u + y_d}{\rho}}. \tag{7.1}$$

Since each asset is in the unit supply, w also equals the asset turnover.

Third, we study marketwide liquidity by looking at the mass of liquid assets $L = |\Theta_L|$ and the average and aggregate costs of bargaining delays for liquid assets $\overline{x} = \frac{1}{L}\int_{\theta \in \Theta_L} x(\theta) dF_L(\theta)$ and $X = \int_{\theta \in \Theta_L} x(\theta) d\theta$, resp. The former captures the marketwide extensive margin, while the latter two capture the marketwide intensive margin. We also look at $W = \frac{\Lambda_s y_d}{y_u + y_d + \Lambda_s} X^{\frac{y_u + y_d}{\rho}}$ as an (imperfect) proxy for the market trade volume.[37] In the analysis, we focus on Θ_L rather than $\Theta_L \cup \Theta_M$ as it allows for a cleaner analytic results, and as our simulations indicate (see the working paper version of this paper), the difference between the two is often small.

Asset Prices and Bid-Ask Spread We next proposition describes the equilibrium bid-ask spread and the asset price decomposition.

Proposition 4. *The bid-ask spread equals $\frac{k}{r}$ and prices of assets are given by*

$$p(\theta) = \underbrace{\frac{1}{r}\left(d + k\theta - (r + y_d)\xi\right) + (1-\alpha)\xi}_{\textit{fundamental value}} + \underbrace{(1-\alpha)\frac{y_d}{r}\frac{\sigma(\theta)\Lambda_s}{\rho + \sigma(\theta)\Lambda_s}\xi x(\theta)}_{\textit{liquidity premium}} - \underbrace{\alpha\frac{y_u}{r}\frac{\Lambda_b}{\rho + \Lambda_b}\xi\overline{x}}_{\textit{average liqudity}}. \tag{7.2}$$

In Duffie et al. (2005), market makers who have access to the inter-dealer market can charge their customers fees for providing immediacy. This results in the endogenous bid-ask spread arising from the search friction. In our model the bid-ask spread arises even

[36] See $(A.18)$ and the characterization of M in the Appendix.

[37] W approximates the aggregate trade volume when r is small relative to $y_u + y_d$ and the set of conditionally liquid assets Θ_M is small.

in the absence of market makers and is determined by the bargaining friction. When k is large, the variation in asset prices is higher and the bargaining over the price starts from offers that are farther apart thus increasing the bid-ask spread.

Equation (7.2) provides an intuitive asset price decomposition. The first component is the price if neither side had an option to search. The other two components reflect the outside options created by the search market. The liquidity premium depends on the costs of bargaining delay $x(\theta)$ and it reflects the seller's outside option. For a more liquid asset, after the seller finds another match less surplus is dissipated due to bargaining delay, which increases the value of the seller's outside option and hence the asset price. This outside option is more valuable when the seller can more easily locate buyers in the market (higher Λ_s). This dependence of the sensitivity of the price to liquidity on aggregate market conditions has been documented empirically (Bao et al. (2011), Friewald et al. (2012)). The third component depends on the average liquidity $\overline{x}$ of liquid assets (in Θ_L) and it reflects the buyer's outside option. When the buyer expects that his next trading partner will have more liquid asset, the value of the buyer's outside option increases which in turn decreases the price. This effect is larger the easier it is for the buyer to find a trade partner (higher Λ_b). Longstaff et al. (2005) show that the non-default component of corporate spreads varies with liquidity measures in the cross-section of assets, but also depends on the marketwide liquidity in the time series analysis, which is in line with the decomposition in equation (7.2).

U-shaped Liquidity We next describe the dependence of the liquidity on the asset quality.

Proposition 5. *The costs of bargaining delays $x(\cdot)$ and trade volume $w(\cdot)$ are U-shaped in quality θ (or the price $p(\theta)$). Moreover, Θ_I is an interval in the interior of* $[0, 1]$.

The U-shaped asset liquidity follows from the bargaining dynamics behind the screening bargaining solution. For the lower qualities, the seller's value of the asset is low and so she prefers to accept a larger discount earlier rather than wait longer for more favorable offers. Symmetrically, the buyer of the higher qualities has a high value and so he prefers to accept a high price early on rather than wait for the lower offers. Qualities in the middle however trade with the longest delay, as for them, investors prefer to wait until either the opponent gives in or reduces his/her demand. Hence, such qualities are less liquid, and in fact, may not be traded at all. For such qualities, the buyer prefers to continue the search rather than start the lengthy negotiation over the price.

The U-shaped prediction is a novel empirical implication and it differs from the decreasing relationship between the liquidity and quality in the asymmetric information models of liquidity (e.g. Guerrieri and Shimer (2014)). When the seller has superior information about the quality (e.g. the bank selling its loans to external investors), to incentivize the seller to reveal her private information, assets of higher quality are traded at higher prices but with lower probability compared to the lower-quality assets. In contrast, our model describes environments where both parties have precise private information about the quality but lack of common knowledge about asset quality which causes the bargaining delay. Let us stress that the U-shaped liquidity holds conditional on the public information about assets. In particular, it does not state that AAA assets and unrated assets are the most liquid, but rather that within the AAA asset class, the liquidity of assets of the highest quality and assets that barely pass the AAA threshold will be the highest.

While in general the liquidity is U-shaped in quality, the shape can be significantly skewed to either side depending on the split of the surplus α as demonstrated in the next proposition.

Proposition 6. *Consider the limit of* $\Theta_I, \Theta_L, x(\cdot)$ *as* $\alpha \to 1$ *denoted by* $\Theta_I^\dagger, \Theta_L^\dagger, x^\dagger(\cdot)$*, resp. Then* $\Theta_I^\dagger = (0, \theta^\dagger)$ *and* $\Theta_L^\dagger = 0 \cup [\theta^\dagger, 1]$ *for some* $\theta^\dagger \in [0, 1)$*, and* $x^\dagger(\cdot)$ *is increasing on* $(0, 1]$.

Higher α increases the costs of delay for the buyer and thus makes him more impatient. As a result, the buyer accepts faster in the bargaining process which in turn increases the liquidity of high-quality assets and expands the region $(\hat{\theta}, 1]$. At the same time, the seller bears a smaller fraction of the delay costs, which increases her incentives to wait longer, and hence, region $[0, \check{\theta})$ shrinks. In the limit when the buyer gets all the surplus, the U-shaped liquidity becomes degenerate: the set of liquid assets consits of some interval of qualities at the top and quality 0. Therefore, we obtain the increasing relationship, as all the trades occur through buyer's acceptance in the SBS.

Market Liquidity We next derive the effect of search and bargaining frictions on the market liquidity.

Proposition 7. *Suppose in equilibrium* $\Theta_I \neq \emptyset$*. Then*

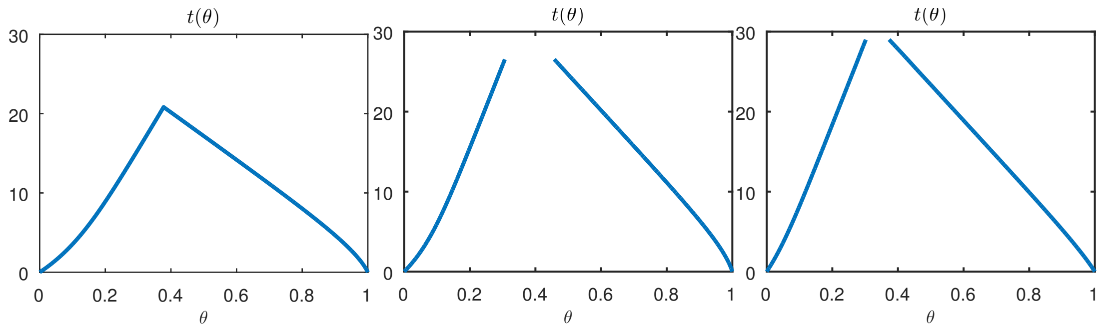
1. *(bargaining friction) for any* $\tilde{L} > 0$*, there exists* $\tilde{\alpha} \in (0, 1)$ *such that if* $\alpha > \tilde{\alpha}$ *and in equilibrium* $L > \tilde{L}$*, then* $\frac{dL}{dk} < 0, \frac{d\underline{x}}{dk} < 0, \frac{d\overline{x}}{dk} < 0, \frac{dX}{dk} < 0, \frac{d\Lambda_s}{dk} > 0$*;*

2. *(search friction)* $\frac{dL}{d\lambda} < 0, \frac{d\underline{x}}{d\lambda} > 0, \frac{d\overline{x}}{d\lambda} > 0, \frac{dX}{d\lambda} < 0, \frac{d\Lambda_s}{d\lambda} > 0.$

To see the effect of the bargaining friction, start with the case $k \approx 0$. Then differences in flow payoffs across assets are very small, and hence, various asset qualities are traded at similar prices. As a result, there are little incentives to delay trade for a slightly more favorable offer and negotiations are short. An increase in the asset heterogeneity k leads to larger differences in prices at which assets are traded. Hence, the negotiation starts from the offers that are farther apart (bid-ask spread increases by Proposition 4) and investors have more incentives to delay trade and wait for a more favorable price offer. The increase in bargaining delays makes fewer assets attractive for trade to buyers and L drops (see Figure 5*b*). However, if buyers did not compromise their required liquidity threshold $\underline{x}$, then few assets will qualify and buyers will end up searching longer for sufficiently liquid assets. Thus, buyers have to decrease their required liquidity threshold $\underline{x}$ and bear part of the increase in the bargaining delays. Note that as the bargaining friction becomes more severe, it becomes easier for sellers to find trade partners (Λ_s increases). This however is accompanied (and explained) by the reduction in the set of acceptable assets and so, leads to a higher match intensity only for sellers of liquid assets. Higher contact intensity for sellers positively affects the trade volume proxy W, however, the increased negotiation times and fewer actively traded assets reduce X and thus, negatively affect W.

An increase in the search friction leads to a wider range of assets actively traded, i.e. L increases (see Figure 5*c*). When it is harder for buyers to find another sellers, the outside option of searching further in the market depreciates and buyers are willing to accept a wider range of assets. This increases the required liquidity threshold $\underline{x}$ and average liquidity of acceptable assets $\overline{x}$, but negatively affects the aggregate liquidity measure X. As a result, more efficient search positively affects aggregate trade volume W through higher contact intensity for sellers Λ_s, but it also makes buyers more selective which is a countervailing force negatively affecting W through X.

One standard critisism of using search delays to explain OTC liquidity is that in many OTC markets, e.g. the corporate bonds market, the common view is that trade delays are not significant and do not have a first-order effect on liquidity. Proposition 7 shows that short trade delays reflect only the intensive margin and can also indicate market illiquidity. Even when trade delays for executed trades are small, this may imply that a vast amount of trades cannot be executed as they are rejected by buyers. In particular, when the search delays are reduced (λ increases), few assets are actively traded (L decreases), while the average bargaining delays are also short ($\overline{x}$ increases). Thus, short *observed* search and bargaining delays do not mean that assets can be quickly sold.



(a) Baseline specification
$\lambda = 1500, k = .01$

(b) High bargaining friction
$\lambda = 1500, k = .015$

(c) High search friction
$\lambda = 100, k = .015$

Figure 5: ***Bargaining delay as a function of asset quality****. Parameters $y_u = 70, y_d = .2, r = 12\%, \alpha = .7, a = 1.5$. Bargaining delay has an inverse U-shaped form, as $x(\theta)$ is U-shaped. The gap in the graph depicts the set of illiquid assets which expands as the bargaining friction increases (from panel (a) to panel (b)) and shrinks as the search friction increases (from panel (b) to panel (c)).*

Market Uncertainty and Transparency The opposite effect on the extensive margin of two frictions again suggests that search delays may not be the right proxy for bargaining delays. We next demonstrate that this difference is important for the effect of market uncertainty on liquidity and the design of OTC markets.

Our model captures the effect of the quality of public information on market liquidity, the dimension that is not present in the current OTC models based on search delays. In the recent financial crisis, credit ratings became less reliable in assessing the quality of assets, such as mortgage-backed securities, collateralized debt obligations, etc., and coincided with the dried-up liquidity of structured finance products. This is consistent with the effect of an increased bargaining friction on market liquidity: the drop in the quality of ratings exacerbates the bargaining friction which in turn results in the reduction in the range of actively traded assets. Notice that more severe search friction leads to the opposite conclusion. While the aggregate trade volume may fall, the range of traded assets expands as the search becomes harder.

The analysis of two trade frictions adds to the debate about the design of OTC markets. On the one hand, such policies as more accurate and frequently updated credit ratings, introduction of benchmarks, standardization of products, dissemination of past quotes, etc., improve the quality of public information. As a result, conditional on better public information, the asset heterogeneity is reduced which decreases the bargaining friction and thus increases the market liquidity L. On the other hand, introduction of more efficient trading platform and greater post-trade transparency reduce the search friction and thus lead to lower market liquidity. Thus, while market changes that improve the search can make risk-sharing more efficient, there are winners and loosers of such measures, as some sellers are not able to liquidate their positions in the more transparent markets. This is consistent with the mixed evidence on effect of post-trade transparency on the liquidity of corporate bonds market (Bessembinder et al. (2006), Edwards et al. (2007), Asquith et al. (2013)).

8 Transparency and Flights-to-Liquidity

This section shows that the substitutability between asset classes leads to flights-to-liquidity during periods of market uncertainty and adverse liquidity effects of the gradual transparency policies.

To analyse these effects, we extend the baseline model to two asset classes. There are two asset classes indexed by $i = 1, 2$, each class is in unit fixed supply. The total mass of investors is $a > 2$. Asset class i is characterized by their asset heterogeneity k_i. The mass $a_i \geq 1$ of investors trading assets in each class i is determined in equilibrium so that $a_1 + a_2 = a$. Other than that, parameters of the model are as in the baseline model. The equilibrium of this two-class extension is defined next (Subscripts indicate equilibrium quantities for the corresponding asset class).

Definition 3. *A tuple $(\sigma_i, M_i, a_i)_{i=1,2}$ is a* two-class equilibrium *if (σ_i, M_i) is the equilibrium of the baseline model with mass of investors a_i and the following conditions hold*

$$\begin{cases} \underline{x}_1 = \underline{x}_2, & \text{if } a - 1 > a_1 > 1, \\ \underline{x}_1 \leq \underline{x}_2, & \text{if } a_1 = 1, \\ \underline{x}_1 \geq \underline{x}_2, & \text{if } a_1 = a - 1. \end{cases} \tag{8.1}$$

Condition (8.1) reflects the mobility of buyers. If trading assets in one of the classes is more profitable, buyers will migrate into trading this asset class. To see this, recall that the buyers' utility of trading each asset class is proportional to strategy thresholds $\underline{x}_1$ and $\underline{x}_2$ (cf. Lemma 2). If both are equal, then buyers are indifferent between trading two classes. If one is greater, then all investors migrate to the more preferable (for buyers) class making the other class illiquid. In the Appendix, we show that the equilibrium of the two-class model exists and is unique. We next show that flights-to-liquidity occur after an increase in the bargaining friction or a decrease in the bargaining friction in the asset substitutes.

Proposition 8. *Suppose $a - 1 > a_1 > 1$ and consider an increase in k_1 such that in the baseline model with only asset class 1 (and mass of investors a_1), it results in a decrease in L_1. Then L_1 and a_1 decrease and the decrease in L_1 is larger in the two-class model. The same conclusion holds if instead λ_2 increases or k_2 decreases so that in the baseline model with only asset class 2 (and mass of investors a_2), L_2 decreases.*

First, when the bargaining friction increases in asset class 1, investors migrate to trading assets in class 2 for which the bargaining friction is lower. This flight-to-liquidity exacerbates the drop in market liquidity L from the increase in the bargaining friction, as buyers' migration improves the ratio of buyers to sellers in the market for class-1 assets and thus allows them to be more selective in their choice of assets for trade. OTC markets are known to be prone to flights-to-liquidity episodes when, due to increased market uncertainty, investors shift their portfolio preferences to safer and more liquid assets. (See Friewald et al. (2012), Dick-Nielsen et al. (2012) for the analysis of flight-to-quality episodes during the recent liquidity crisis of 2007-2008). Flights-to-liquidity are associated with dried-up liquidity in markets for more risky assets which is consistent with the predictions of our model. An important observation is that the level of payoffs d in each asset class does not affect the liquidity implications of our model. Thus, we stress that the flights are flights-to-liquidity rather than flights-to-quality which is consistent with the empirical evidence that default risk plays a smaller role than liquidity in flights (see Beber et al. (2009)).

Now, consider the effect of a decrease in the bargaining friction in class 2 on the class 1 liquidity. Because of the lower bargaining delays, the substitute asset class 2 becomes more attractive. In order to maintain the indifference of buyers between two classes, some buyers migrate into trading class 2 which improves the ratio of buyers to sellers in the class 1 and hence the profit of buyers from trading class 1 assets. The effect on the market liquidity L is similar to the effect of the reduction in the search friction in the baseline model. As class 2 becomes more attractive, buyers become more selective in assets they accept for trade in class 1 and so, the reduce the range of actively traded class-1 assets. In fact, the reduction in the search friction in class 2 would have the same effect on the liquidity of asset class 1.

This result shows that a gradualism in introducing transparency or improving the quality of public information can hurt the market liquidity. E.g. the post-trade transparency in the corporate bonds market was introduced starting in 2002 in several phases with early phases requiring disclosure only for larger issues of investment grade bonds, and later phases expanding the requirement to high-yield bonds and other assets, such as agency-backed securities. We have already discussed that the reduction in the search friction can reduce the market liquidity L of the asset class affected, Propositionit also has a negative liquidity on the liquidity of substitute asset classes. Asquith et al. (2013) shows that the introduction of post-trade transparency in corporate bonds market while decreasing the price dispersion, also decreased the trading activity in high-yield bonds which started before the post-trade transparency was actually implemented for such bonds.

9 Conclusion

This paper develops a tractable model of decentralized asset markets with both search and endogenous bargaining delays. The key to the tractability is the application of the novel screening bargaining solution that captures bargaining delays due to the gap between the coarse public and precise private information. The analysis allows for the separation between the intensive and extensive trade margins, as well as between the search and bargaining frictions. The liquidity of the asset is U-shaped in the quality and assets in the middle of the quality range may not be traded at all. While on the intensive margin, search and bargaining frictions operate similarly, on the extensive margin, they are quite different. The search friction increases, while the bargaining friction decreases the market liquidity. This shows that greater transparency can hurt liquidity if it leads to lower search frictions in the market. we also show that because of the substitutability of asset classes, the OTC markets are prone to flights-to-liquidity and gradualism in the introduction of transparency can have adverse effects for the market liquidity. Finally, we derive the asset price decomposition which holds for a variety of specifications of post-search delay.

A Appendix

Appendix $A.1$ contains the proofs for the micro-foundations of the SBS. Appendix $A.2$ contains the the analysis of the OTC model. The Online Appendix contains proofs of the auxiliary results.

A.1 Microfoundation for the Screening Bargaining Solution

Proof of Part 1 of Theorem 1

Denote $\overline{\alpha} = \frac{1-e^{-\rho\Delta_b}}{1-e^{-\rho\Delta}}$, $\underline{\alpha} = \frac{e^{-\rho\Delta_s}-e^{-\rho\Delta}}{1-e^{-\rho\Delta}}$, $\overline{\mathbf{p}}(\theta^s,\theta^b) = (1-\underline{\alpha})v(\theta^b) + \underline{\alpha}c(\theta^s)$, and $\underline{\mathbf{p}}(\theta^s,\theta^b) = (1-\overline{\alpha})v(\theta^b) + \overline{\alpha}c(\theta^s)$. Let $\underline{\ell}$ and $\overline{\ell}$, resp., be the minimum and maximum on [0,1] of derivatives of v. We first derive the following bounds on the price of trade.

Lemma 4. *Suppose after some history, the highest remaining types of the buyer and seller equal to $\hat{\theta}^b$ and $\hat{\theta}^s$, resp., and the lowest remaining types of the buyer and seller equal to $\check{\theta}^b$ and $\check{\theta}^s$, resp. Suppose $\hat{\theta}^b > \check{\theta}^b$ and $\hat{\theta}^s > \check{\theta}^s$. Then*

$$\underline{\mathbf{p}}(\check{\theta}^s,\check{\theta}^b) \le p_{F,\Delta}(\theta^s,\theta^b) \le \overline{\mathbf{p}}(\hat{\theta}^s,\hat{\theta}^b). \tag{A.1}$$

Moreover, any offer below $\underline{\mathbf{p}}(\check{\theta}^s,\check{\theta}^b)$ and any offer above $\overline{\mathbf{p}}(\hat{\theta}^s,\hat{\theta}^b)$ is accepted with probability 1 by the buyer and seller, resp.

Proof. Let P, resp. Q, be the supremum over all histories of price offers accepted by the buyer, resp. rejected by the seller, with positive probability in PBE. By the definition of Q, after any history the buyer's type θ^b can guarantee himself the utility arbitrarily close to $e^{-\rho\Delta_s}(v(\theta^b)-Q)$ by making an offer marginally above Q (that is guaranteed to be accepted by the seller). By the definition of P,

$$e^{-\rho\Delta_s}(v(\theta^b)-Q) \le v(\theta^b) - P. \tag{A.2}$$

Let $U(\theta^s)$ be the supremum over all histories of the continuation utilities of the seller's type θ^s after the rejection of the offer in the current round. If type θ^s rejects an offer, she cannot guarantee more than $\max\{e^{-\rho\Delta_b}(P-c(\theta^s)), e^{-\rho\Delta}U(\theta^s)\}$, which implies $U(\theta^s) \le e^{-\rho\Delta_b}(P-c(\theta^s))$. By the definition of Q,

$$Q - c(\theta^s) \le e^{-\rho\Delta_b}(P - c(\theta^s)). \tag{A.3}$$

By $(A.3)$, $Q \le P$. Combining $(A.2)$ and $(A.3)$, we get

$$\begin{aligned} P &\le (1-e^{-\rho\Delta_s})v(\overline{\theta}^b) + e^{-\rho\Delta_s}Q \\ &\le (1-e^{-\rho\Delta_s})v(\overline{\theta}^b) + e^{-\rho\Delta_s}(1-e^{-\rho\Delta_b})c(\overline{\theta}^s) + e^{-\rho\Delta}P, \end{aligned} \tag{A.4}$$

where we used the fact that the support of beliefs does not expand to put an upper bound on $v(\theta^b)$ and $c(\theta^s)$ in $(A.2)$ and $(A.3)$. By iterating the inequality $(A.4)$, we obtain the upper bound in $(A.1)$. By the definition of Q, any offer above $\overline{\mathbf{p}}(\hat{\theta}^s,\hat{\theta}^b)$ is accepted with probability one by the seller. The argument for the lower bound is symmetric. □

Denote $D = \{(\theta^s,\theta^b) : |\theta^s-\theta^b| < \frac{1}{4}\varepsilon^2\}$, $\Omega = \{(\theta^s,\theta^b) : |p_{F,\Delta}(\theta^s,\theta^b) - \mathbf{p}(\theta^s,\theta^b)| < \frac{1}{4}\varepsilon^2\}$, and $\Theta = \Omega \cap D$. Fix $\varepsilon > 0$. For F sufficiently far in the sequence, $\mathbb{P}_F(D) > 1 - \frac{1}{4}\varepsilon^2$. Moreover, for any such F and Δ sufficiently small, there is a PBE in $\mathcal{G}(F, \Delta)$ such that $\mathbb{P}_F(\Omega) > 1 - \frac{1}{4}\varepsilon^2$. Define $B^{1-\varepsilon,s} = \{\theta^s : \mathbb{P}_F(\Theta|\theta^s) > 1 - \varepsilon\}$ and $B^{1-\varepsilon,b} = \{\theta^b : \mathbb{P}_F(\Theta|\theta^b) > 1 - \varepsilon\}$.

Lemma 5. *For any interval I such that $|I| < \varepsilon$, $I \cap B^{1-\varepsilon,s} \neq \emptyset$ and $I \cap B^{1-\varepsilon,b} \neq \emptyset$.*

Proof. We show that $I \cap B^{1-\varepsilon,s} \neq \emptyset$ ($I \cap B^{1-\varepsilon,b} \neq \emptyset$ is analogous). Let F^s be the marginal of F on θ^s. First, we show that $\mathbb{P}_{F^s}(B^{1-\varepsilon,s}) > 1 - \frac{\varepsilon}{2}$. Note that

$$\mathbb{P}_F(\Theta) = \mathbb{P}_F(\Omega) + \mathbb{P}_F(D) - \mathbb{P}_F(\Omega \cup D) \geq \mathbb{P}_F(\Omega) + \mathbb{P}_F(D) - 1 > 1 - \frac{\varepsilon^2}{2}.$$

Now,

$$\mathbb{P}_F(\Theta) = \int_0^1 \mathbb{P}_F(\Theta|\theta^s) dF^s(\theta^s) \leq (1-\varepsilon)(1 - \mathbb{P}_{F^s}(B^{1-\varepsilon,s})) + \mathbb{P}_{F^s}(B^{1-\varepsilon,s}) = 1 - \varepsilon + \varepsilon \mathbb{P}_{F^s}(B^{1-\varepsilon,s})$$

and so, $\mathbb{P}_{F^s}(B^{1-\varepsilon,s}) \geq 1 - \frac{1}{\varepsilon}(1 - \mathbb{P}_F(\Theta)) > 1 - \frac{\varepsilon}{2}$.

Note that $F \xrightarrow{p} F^* \implies F \xrightarrow{d} F^* \implies F^s \xrightarrow{d} F^{s*}$. Since F^s and F^{s*} are continuous, they converge uniformly as functions of θ^s. Thus, for F far enough in the sequence $|F^s(\theta^s) - F^{s*}(\theta^s)| < \frac{\varepsilon}{4}$ for all θ^s. Let $I = [\check{\theta}^s, \hat{\theta}^s]$. By the triangular inequality,

$$|\mathbb{P}_{F^s}(I) - \mathbb{P}_{F^{s*}}(I)| \leq |F^s(\hat{\theta}^s) - F^{s*}(\hat{\theta}^s)| + |F^s(\check{\theta}^s) - F^{s*}(\check{\theta}^s)| \leq \frac{\varepsilon}{2},$$

and so, $\mathbb{P}_{F^s}(I) \in [|I| - \frac{\varepsilon}{2}, |I| + \frac{\varepsilon}{2}]$. Therefore, $\mathbb{P}_{F^s}(I \cap B^{1-\varepsilon,s}) \geq \mathbb{P}_{F^s}(I) + \mathbb{P}_{F^s}(B^{1-\varepsilon,s}) - 1 \geq |I| - \varepsilon > 0$ which proves $I \cap B^{1-\varepsilon,s} \neq \emptyset$. □

Let $[\underline{s}_0^*, \overline{s}_0^*] = [\underline{b}_0^*, \overline{b}_0^*] = [0, 1]$. For $n = 1, \ldots, N$ (N to be specified below), define the nested intervals of seller's types $[\underline{s}_n^*, \overline{s}_n^*)$ as follows. For given $[\underline{s}_{n-1}^*, \overline{s}_{n-1}^*)$, let $\mathcal{S}_n^*$ be the collection of all intervals $[\underline{s}_{n-1}, \overline{s}_{n-1})$ such that

- before round n, seller's types in $[\underline{s}_n, \overline{s}_n)$ pool with types in $[\underline{s}_{n-1}^*, \overline{s}_{n-1}^*)$, and in round n, they pool with each other and separate from types in $[\underline{s}_{n-1}^*, \overline{s}_{n-1}^*) \backslash [\underline{s}_n, \overline{s}_n)$, and (since players use interval strategies, $[\underline{s}_n, \overline{s}_n)$ is well defined);

- $\underline{s}_n < 1 - 2\varepsilon$.

Let $[\underline{s}_n^*, \overline{s}_n^*)$ be the set in $\mathcal{S}_n^*$ such that $\underline{s}_n^* > \underline{s}_n$ for all intervals $[\underline{s}_n, \overline{s}_n) \in \mathcal{S}_n^*$. Analogously, for $n = 1, \ldots, N$, define the nested intervals of buyer's types $(\underline{b}_n^*, \overline{b}_n^*]$ as follows. For given $(\underline{b}_{n-1}^*, \overline{b}_{n-1}^*]$, let $\mathcal{B}_n^*$ be the collection of all $(\underline{b}_n, \overline{b}_n]$ of seller's types that satisfy

- buyer's types in $(\underline{b}_n, \overline{b}_n]$ pool with each other in round n and pool with types in $(\underline{b}_{n-1}^*, \overline{b}_{n-1}^*]$ before round n;

- $\overline{b}_n > 2\varepsilon$.

Let $(\underline{b}_n^*, \overline{b}_n^*]$ be the set in $\mathcal{B}_n^*$ such that $\overline{b}_n^* < \overline{b}_n$ for all $(\underline{b}_n, \overline{b}_n] \in \mathcal{B}_n^*$. Let round N be the first round n in which either $\frac{1}{3} \le \underline{s}_n^*$ or $\overline{b}_n^* \le \frac{2}{3}$.

Lemma 6. *For ε sufficiently small and any $n < N$,*

1. $\overline{s}_n^* \ge 1 - 2\varepsilon$ *and* $\underline{b}_n^* \le 2\varepsilon$;
2. $\overline{s}_n^* - \underline{s}_n^* > \frac{1}{3}$ *and* $\overline{b}_n^* - \underline{b}_n^* > \frac{1}{3}$;
3. *types in* $(\underline{s}_n^*, \overline{s}_n^*]$ *and* $(\underline{b}_n^*, \overline{b}_n^*]$ *reject the opponent's offer and make some counter-offers* p_n^{s*} *and* p_n^{b*}, *resp.;*
4. *there is a positive constant C (independent of n) such that*

$$p_n^{b*} \le \mathbf{p}(\tfrac{1}{3}) + C\varepsilon; \tag{A.5}$$

$$p_n^{s*} \ge \mathbf{p}(\tfrac{2}{3}) - C\varepsilon. \tag{A.6}$$

Proof. 1,2) Fix $n < N$. If $\overline{s}_n^* < 1 - 2\varepsilon$, then there is $[\underline{s}_n, \overline{s}_n) \in \mathcal{S}_n^*$ such that $\underline{s}_n = \overline{s}_n^* > \underline{s}_n^*$ which contradicts the definition of $[\underline{s}_n^*, \overline{s}_n^*)$. Thus, $\overline{s}_n^* \ge 1 - 2\varepsilon$, and analogously, $\underline{b}_n^* \le 2\varepsilon$. Since $\underline{s}_n^* < \frac{1}{3}$ by the definition of N, $\overline{s}_n^* - \underline{s}_n^* \ge \frac{2}{3} - 2\varepsilon > \frac{1}{3}$ (when $\varepsilon < \frac{1}{6}$), and analogously, $\overline{b}_n^* - \underline{b}_n^* > \frac{1}{3}$.

3) Suppose before round n, both players pool with $[\underline{s}_n^*, \overline{s}_n^*)$ and $(\underline{b}_{n-1}^*, \overline{b}_{n-1}^*]$, resp., and suppose to contradiction that seller's types in $[\underline{s}_n^*, \overline{s}_n^*)$ accept p_{n-1}^{b*}. Let $\hat{\theta}^s = \sup\{(\frac{1}{3}, \frac{2}{3} - \frac{1}{4}\varepsilon^2) \cap B^{1-\varepsilon,s}\}$ and $\check{\theta}^s = \inf\{(\frac{1}{3}, \frac{2}{3} - \frac{1}{4}\varepsilon^2) \cap B^{1-\varepsilon,s}\}$ (these types also accept p_{n-1}^{b*} as $[\check{\theta}^s, \hat{\theta}^s] \subseteq [\underline{s}_n^*, \overline{s}_n^*)$). We first show that for some c_0, $|\mathbf{p}(\theta^s) - p_{n-1}^{b*}| \le c_0\varepsilon$ for $\theta^s \in \{\check{\theta}^s, \hat{\theta}^s\}$. Consider $\theta^s \in \{\check{\theta}^s, \hat{\theta}^s\}$. Since $\theta^s \in B^{1-\varepsilon,s}$, type θ^s assigns probability at least $1-\varepsilon$ to Θ. Note that $[\theta^s - \frac{1}{4}\varepsilon^2, \theta^s + \frac{1}{4}\varepsilon^2] \subseteq [2\varepsilon, \frac{2}{3}] \subseteq (\underline{b}_{n-1}^*, \overline{b}_{n-1}^*]$, and so after round $n-1$ the probability of Θ is at least $\frac{1-\varepsilon}{1-\beta}$ where $\beta = 1 - F(\overline{b}_{n-1}^* | \theta^s) + F(\underline{b}_{n-1}^* | \theta^s)$. Since type θ^s accepts p_{n-1}^{b*} and $\theta^s \in B^{1-\varepsilon,s}$, it is necessary

$$\begin{aligned} p_{n-1}^{b*} \in \Big[&\tfrac{1-\varepsilon}{1-\beta}(\mathbf{p}(\theta^s - \tfrac{1}{4}\varepsilon^2) - \tfrac{1}{4}\varepsilon^2) + \tfrac{\varepsilon-\beta}{1-\beta}\underline{\mathbf{p}}(0), \tfrac{1-\varepsilon}{1-\beta}(\mathbf{p}(\theta^s + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2) + \tfrac{\varepsilon-\beta}{1-\beta}\overline{\mathbf{p}}(1)\Big] \subseteq \\ &\big[(1-\varepsilon)(\mathbf{p}(\theta^s - \tfrac{1}{4}\varepsilon^2) - \tfrac{1}{4}\varepsilon^2) + \varepsilon\underline{\mathbf{p}}(0), (1-\varepsilon)(\mathbf{p}(\theta^s + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2) + \varepsilon\overline{\mathbf{p}}(1)\big] \end{aligned} \tag{A.7}$$

and so, $|\mathbf{p}(\theta^s) - p_{n-1}^{b*}| \le c_0\varepsilon$ where c_0 is some positive constant. Thus,

$$\underline{\ell}(\hat{\theta}^s - \check{\theta}^s) \le \mathbf{p}(\hat{\theta}^s) - \mathbf{p}(\check{\theta}^s) \le |\mathbf{p}(\check{\theta}^s) - p_{n-1}^{b*}| + |\mathbf{p}(\hat{\theta}^s) - p_{n-1}^{b*}| \le 2c_0\varepsilon$$

and so, $\hat{\theta}^s - \check{\theta}^s \le \frac{2c_0}{\underline{\ell}}\varepsilon$. On the other hand, by Lemma 5, $\hat{\theta}^s - \check{\theta}^s \ge \frac{1}{3} - 3\varepsilon$ and so, for $\varepsilon < \frac{\underline{\ell}}{3(3\underline{\ell}+2c_0)}$ this leads to a contradiction. Therefore, types in $[\underline{s}_n^*, \overline{s}_n^*)$ reject p_{n-1}^{b*} and make a counter-offer p_n^{s*}. The argument is analogous for the buyer.

4) Consider type $\check{\theta}^s$ defined above. By Lemma 5, $\check{\theta}^s < \frac{1}{3} + \varepsilon$. By $(A.7)$,

$$p_{n-1}^{b*} \le (1-\varepsilon)(\mathbf{p}(\check{\theta}^s + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2) + \varepsilon\overline{\mathbf{p}}(1) \le (1-\varepsilon)(\mathbf{p}(\tfrac{1}{3} + \varepsilon + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2) + \varepsilon\overline{\mathbf{p}}(1) \le \mathbf{p}(\tfrac{1}{3}) + C\varepsilon,$$

where C is some constant. $\square$

Lemma 7. *For sufficiently small ε, one of the two obtains:*

- *there is the seller's type $\tilde{\theta}^s \in (0, 4\varepsilon) \cap B^{1-\varepsilon,s}$ and the seller's strategy that guarantees the expected utility at least $(1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\frac{1}{3}, 0) - c(\tilde{\theta}^s))$ in the beginning of the game;*
- *there is the buyer's type $\tilde{\theta}^b \in (1-4\varepsilon, 1) \cap B^{1-\varepsilon,b}$ and the buyer's strategy that guarantees the expected utility at least $(1-\varepsilon)e^{-\rho\Delta(N+1)}(v(\tilde{\theta}^b) - \overline{\mathbf{p}}(1, \frac{2}{3}))$ in the beginning of the game.*

Proof. There are two cases depending on whether $\frac{1}{3} \leq \underline{s}^*_N$ or $\overline{b}^*_N \leq \frac{2}{3}$ and $\frac{1}{3} > \underline{s}^*_N$

Case 1: $\frac{1}{3} \leq \underline{s}^*_N$. There are two possibilities:

1. First, suppose that there is $\eta \leq N$ and an interval of seller's types $[\underline{s}_\eta, \overline{s}_\eta) \subset [\underline{s}^*_{\eta-1}, \overline{s}^*_{\eta-1})$ with $\underline{s}^*_N \leq \underline{s}_\eta$ that reject $p^{b*}_{\eta-1}$ and make a counter-offer $\hat{p} > \underline{\mathbf{p}}(\underline{s}^*_N, 0)$ in round η. By Lemma 4, if such a counter-offer in round η is rejected, then the seller can guarantee to trade at price $\underline{\mathbf{p}}(\underline{s}_\eta, 0)$ in round $\eta+1$. Consider the following strategy of the seller:

   - as long as the buyer pools with types in $[\underline{b}^*_{n-1}, \overline{b}^*_{n-1})$, the seller pools with types $[\underline{s}^*_n, \overline{s}^*_n)$ for $n \leq \eta$, pools with $[\underline{s}_\eta, \overline{s}_\eta)$ in round η and, if in round n, offer $\hat{p}$ is rejected, offers $\underline{\mathbf{p}}(\underline{s}^*_\eta, 0)$ in round $\eta+1$;
   - otherwise, the seller rejects all offers and makes unacceptable offers above $\overline{\mathbf{p}}(1)$.

   Type $\tilde{\theta}^s \in (2\varepsilon + \frac{1}{4}\varepsilon^2, 4\varepsilon) \cap B^{1-\varepsilon,s}$ (this set is non-empty by Lemma 5) assigns probability at least $1-\varepsilon$ to the buyer's type belonging in $[\underline{b}^*_{\eta-1}, \overline{b}^*_{\eta-1})$, as $[\tilde{\theta}^s - \frac{1}{4}\varepsilon^2, \tilde{\theta}^s + \frac{1}{4}\varepsilon^2] \subseteq [2\varepsilon, \frac{2}{3}] \subseteq [\underline{b}^*_{\eta-1}, \overline{b}^*_{\eta-1})$. Therefore, by deviating to the described strategy, type $\tilde{\theta}^s$ can guarantee utility

   $$\begin{aligned}(1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\underline{s}_\eta, 0) - c(\tilde{\theta}^s)) &\geq (1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\underline{s}^*_N, 0) - c(\tilde{\theta}^s)) \\ &\geq (1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\tfrac{1}{3}, 0) - c(\tilde{\theta}^s)).\end{aligned}$$

2. Now, suppose that for any $\eta \leq N$ and any interval of seller's types $[\underline{s}, \overline{s}) \subset [\underline{s}^*_{\eta-1}, \overline{s}^*_{\eta-1})$ with $\underline{s}^*_N \leq \underline{s}$ that pool with each other and separate from other types in $[\underline{s}^*_{\eta-1}, \overline{s}^*_{\eta-1})$, they either accept $p^{b*}_{\eta-1}$ or make a counter-offer below $\underline{\mathbf{p}}(\underline{s}^*_N, 0)$ in round η. Consider the following strategy of the buyer:

   - for $\eta \leq N$, as long as the seller pools with types in $[\underline{s}^*_\eta, \overline{s}^*_\eta)$, the buyer pools with types in $[\underline{b}^*_N, \overline{b}^*_N)$, unless the seller makes an offer below $\underline{\mathbf{p}}(\underline{s}^*_N, 0)$, in which case the buyer accepts it;
   - otherwise, the buyer rejects all offers and makes unacceptable offers below $\underline{\mathbf{p}}(0)$.

By Lemma 5, there exists type $\tilde{\theta}^b \in (1-2\varepsilon+\frac{1}{4}\varepsilon^2, 1-\frac{1}{4}\varepsilon^2] \cap B^{1-\varepsilon,b}$ that assigns probability at least $1-\varepsilon$ to $\theta^s > 1-2\varepsilon$, as $[\tilde{\theta}^b - \frac{1}{4}\varepsilon^2, \tilde{\theta}^b + \frac{1}{4}\varepsilon^2] \subseteq [1-2\varepsilon, 1] \subseteq [\underline{s}_N^*, 1]$. Therefore, by deviating to the described strategy, type $\tilde{\theta}^b$ can guarantee utility

$$(1-\varepsilon)e^{-\rho\Delta(N+1)}(v(\tilde{\theta}^b) - \max\{p_1^{b*}, \dots, p_{N-1}^{b*}, \underline{\mathbf{p}}(\underline{s}_N^*, 0)\}) \geq \\ (1-\varepsilon)e^{-\rho\Delta(N+1)}(v(\tilde{\theta}^b) - \max\{\mathbf{p}(\tfrac{1}{3}) + C\varepsilon, \underline{\mathbf{p}}(\underline{s}_N^*, 0)\}),$$

where the inequality follows from Lemma 6.

Case 2: $\overline{b}_N^* \leq \frac{2}{3}$, but $\frac{1}{3} > \underline{s}_N^*$. By the symmetric argument as in case 1, one of the following holds:

- there is the seller's type $\tilde{\theta}^s \in (0, 4\varepsilon) \cap B^{1-\varepsilon,s}$ and the seller's strategy that guarantees the expected utility at least $(1-\varepsilon)e^{-\rho\Delta(N+1)}(\min\{\mathbf{p}(\frac{2}{3}) - C\varepsilon, \overline{\mathbf{p}}(1, \overline{b}_N^*)\} - c(\tilde{\theta}^s))$ in the beginning of the game;
- there is the buyer's type $\tilde{\theta}^b \in (1-4\varepsilon, 1) \cap B^{1-\varepsilon,b}$ and the buyer's strategy that guarantees the expected utility at least $(1-\varepsilon)e^{-\rho\Delta(N+1)}(v(\tilde{\theta}^b) - \overline{\mathbf{p}}(1, \frac{2}{3}))$ in the beginning of the game.

This completes the proof of the lemma. □

We can now prove part 1 of Theorem 1. Suppose in Lemma 7 the first case is realized (the argument is symmetric for the second case): there is a type $\tilde{\theta}^s \in (0, 4\varepsilon) \cap B^{1-\varepsilon,s}$ who is guaranteed utility $(1-\varepsilon)e^{-r\Delta(N+1)}(\underline{\mathbf{p}}(\frac{1}{3}, 0) - c(\tilde{\theta}^s))$ in the beginning of the game. Since $\tilde{\theta}^s \in B^{1-\varepsilon,s}$, it is necessary that

$$(1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\tfrac{1}{3}, 0) - c(\tilde{\theta}^s)) \leq (1-\varepsilon)(\mathbf{p}(\tilde{\theta}^s + \tfrac{1}{4}\varepsilon^2) - c(\tilde{\theta}^s)) + \varepsilon v(1)$$

and so,

$$e^{-\rho\Delta N} \leq e^{r\Delta} \frac{(1-\varepsilon)(\mathbf{p}(\tilde{\theta}^s + \frac{1}{4}\varepsilon^2) - c(\tilde{\theta}^s)) + \varepsilon v(1)}{(1-\varepsilon)(\underline{\mathbf{p}}(\frac{1}{3}, 0) - c(\tilde{\theta}^s))} \underset{\varepsilon \to 0}{\to} \frac{\mathbf{p}(0) - c(0)}{\underline{\mathbf{p}}(\frac{1}{3}, 0) - c(0)} < 1.$$

Therefore, when ε is small, there is $\overline{T} > 0$ such that $\Delta N > \overline{T}$ and so, types in $(\frac{1}{3}, \frac{2}{3})^2$ trade with a delay at least $\overline{T}$. The probability of such types approaches $\frac{1}{3}$ as $F \xrightarrow{p} F^*$, and so there is $c_1 > 0$ such that when $\varepsilon < \frac{1}{6c_1}$,

$$\mathbb{E}_F[e^{-\rho t_{F,\Delta}}] \leq (\tfrac{1}{3} + c_1\varepsilon)e^{-\rho\overline{T}} \leq \tfrac{1}{2}e^{-\rho\overline{T}} \equiv x_l,$$

for all F sufficiently far in the sequence and all Δ sufficienly small. This proves (4.1).

To show (4.2), observe that every seller type can trade at price $\underline{\mathbf{p}}(0)$. Thus, for any $\theta^s \in B^{1-\varepsilon,s}$,

$$\begin{aligned}\underline{\mathbf{p}}(0) - c(\theta^s) &\leq (1-\varepsilon)\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta](\mathbf{p}(\theta^s + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2 - c(\theta^s)) + \varepsilon v(1) \\ &\leq \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta](1-\alpha)\xi + c_2\varepsilon,\end{aligned}$$

for some constant $c_2 > 0$. This implies that for any $\overline{\theta}^s$

$$\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta] \geq \frac{\underline{\mathbf{p}}(0) - c(\theta^s) - c_2\varepsilon}{(1-\alpha)\xi} > \frac{\underline{\mathbf{p}}(0) - c(\underline{\theta}^s) - c_2\varepsilon}{(1-\alpha)\xi}$$

for all $\theta^s \in B^{1-\varepsilon,s}$ below $\underline{\theta}^s$. By choosing $\underline{\theta}^s$ and ε sufficiently close to zero, we can guarantee that there is $\tilde{x} > x_l$ such that for all Δ sufficiently small, $\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta] \geq \tilde{x}$. Moreover,

$$\begin{aligned}\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s < \underline{\theta}^s] &= \int_0^{\underline{\theta}^s} \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s] dF^s(\theta^s) \\ &\geq \int_0^{\underline{\theta}^s} \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta]\mathbb{P}_F(\Theta|\theta^s) dF^s(\theta^s) \\ &\geq \tilde{x}\int_0^{\underline{\theta}^s} \mathbb{P}_F(\Theta|\theta^s) dF^s(\theta^s) \\ &\geq (1-\varepsilon)\tilde{x}\mathbb{P}_{F^s}(B^{1-\varepsilon,s}) \geq (1-\varepsilon)(1-\tfrac{\varepsilon}{2})\tilde{x}.\end{aligned}$$

Thus, there is $x_h > x_l$ such that for F sufficiently close to F^* and all Δ sufficiently small, $\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s < \underline{\theta}^s] \geq x_h$.

The analogous argument applied to buyer's types close to 1 gives that there is $\overline{\theta}^s > \underline{\theta}^s$ such that $\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^b < \overline{\theta}^s] \geq x_h$. Observing that for fixed $\overline{\theta}^s$, $|\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^b < \overline{\theta}^s] - \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s < \overline{\theta}^s]| \to 0$ as $F \xrightarrow{p} F^*$, we get the inequality (4.2).

Proof of Part 2 of Theorem 1

The proof proceeds as follows. We first introduce and analyze the continuous-time bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot|F)$ which is a generalization of the game $\mathcal{G}(p^s_\cdot, p^b_\cdot)$ in Section 3 to affiliated distributions of types F (thus, $\mathcal{G}(p^s_\cdot, p^b_\cdot) = \mathcal{G}(p^s_\cdot, p^b_\cdot|F^*)$). We then proceed with a series of approximations. First, we approximate the SBS outcome with the BNE outcome in $\mathcal{G}(\cdot|F^*)$. Second we approximate each BNE outcome of $\mathcal{G}(\cdot|F^*)$ with BNE outcomes of $\mathcal{G}(\cdot|F), F \xrightarrow{p} F^*$. Finally, we approximate each BNE outcome of $\mathcal{G}(p^s_\cdot, p^b_\cdot|F)$ with PBE outcome in the discrete-time bargaining game $\mathcal{G}(F, \Delta), \Delta \to 0$. Thus, we construct a sequence of PBE frequent-offer limits (indexed by $F \xrightarrow{p} F^*$) that approximates the SBS outcome, and hence prove the theorem.

Continuous-Time Bargaining Game $\mathcal{G}(\cdot|F)$ Consider a strictly decreasing function $p^s_\cdot$ and a strictly increasing function $p^b_\cdot$ and the following continuous-time bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot|F)$. The buyer continuously makes offers p^b_t and the seller continuously makes offers p^s_t. Players can choose only the time when they accept the price offer of the opponent, and the trade happens once the first acceptance happens (at the accepted price). The difference from $\mathcal{G}(p^s_\cdot, p^b_\cdot)$ is that players' types are distributed according to a general affiliated distribution F, not F^*. We consider BNEs in threshold strategies. Let T be the first time when $p^s_t = p^b_t$. For any $t \in [0, T]$, let θ^s_t and θ^b_t be, resp., strictly increasing and strictly decreasing functions such that $\theta^s_0 = 0$ and $\theta^b_0 = 1$. At time t, all types of the seller below θ^s_t (resp.,, all types of the buyer above θ^b_t) accept the offer p^b_t (resp., p^s_t).

Lemma 8. *Suppose that a tuple $(p^s_\cdot, p^b_\cdot, \theta^s_\cdot, \theta^b_\cdot)$ satisfies the system of differential equations*

$$\begin{aligned} r(v(\theta^b_t) - p^s_t) + \dot{p}^s_t =& (p^s_t - p^b_t)\dot{\theta}^b_t \frac{f(\theta^s_t|\theta^b_t)}{1 - F(\theta^s_t|\theta^b_t)}, \\ -r(p^b_t - c(\theta^s_t)) + \dot{p}^b_t =& (p^s_t - p^b_t)\dot{\theta}^b_t \frac{f(\theta^b_t|\theta^s_t)}{F(\theta^b_t|\theta^s_t)}; \end{aligned} \tag{A.8}$$

with initial conditions $\theta^s_0 = 0$ and $\theta^b_0 = 1$, and $\theta^s_T < 1$ and $\theta^b_T > 0$. Then threshold strategies $(\theta^s_\cdot, \theta^b_\cdot)$ constitute a BNE in $\mathcal{G}(p^s_\cdot, p^b_\cdot|F)$.

Proof. We show that if $\theta^b_\cdot$ satisfies the first equation in the system $(A.8)$, then it is a best response to the threshold strategy $\theta^s_\cdot$. Buyer's type θ^b chooses the acceptance time t to maximize $u(\theta^b, t)$ given by

$$u(\theta^b, t) = \int_0^t e^{-ru}(v(\theta^b) - p^b_u) dF(\theta^s_u|\theta^b) + (1 - F(\theta^s_t|\theta^b))e^{-rt}(v(\theta^b) - p^s_t).$$

The first-order condition for this problem is

$$(p^s_t - p^b_t) f(\theta^s_t|\theta^b)\dot{\theta}^s_t = (1 - F(\theta^s_t|\theta^b))(r(v(\theta^b) - p^s_t) + \dot{p}^s_t).$$

From the first-order condition,

$$\begin{aligned} u(1, t(1)) - u(\tilde{\theta}^b, t(\tilde{\theta}^b)) &= \int_{\tilde{\theta}^b}^1 \left(\frac{\partial}{\partial \theta^b} u(\theta^b, t(\theta^b)) + \frac{\partial}{\partial t} u(\theta^b, t(\theta^b)) t'(\theta^b) \right) d\theta^b \\ &= \int_{\tilde{\theta}^b}^1 \frac{\partial}{\partial \theta^b} u(\theta^b, t(\theta^b)) d\theta^b, \end{aligned} \tag{A.9}$$

where $t(\theta^b)$ is the inverse of $\theta^b_\cdot$. In Claim 1 below, we show that $u(\theta^b, t)$ satisfies the smooth single crossing difference (SSCD) condition in $(\theta^b, -t)$. Together with the envelope formula $(A.9)$, this verifies the conditions of Theorem 4.2 in Milgrom (2004) and proves that $\theta^b_\cdot$ is a best response to $\theta^s_\cdot$. Therefore, $(\theta^s_\cdot, \theta^b_\cdot)$ constitute a BNE of $\mathcal{G}(F)$.

Claim 1. *$u(\theta^b, t)$ satisfies the SSCD condition in $(\theta^b, -t)$.*

Proof: We will show the following conditions are satisfied which imply the SSCD.

1. $u(\theta^b, t)$ satisfies the (strict) single crossing difference condition in $(\theta^b, -t)$, i.e. for all $\tilde{t} > t$ and $\tilde{\theta}^b > \theta^b$,

$$u(\theta^b, t) - u(\theta^b, \tilde{t}) \geq 0 \quad \Longrightarrow \quad u(\tilde{\theta}^b, t) - u(\tilde{\theta}^b, \tilde{t}) > 0.$$

2. for all t, if $\frac{\partial}{\partial t} u(\theta^b, t) = 0$, then for all $\delta > 0$, $\frac{\partial}{\partial t} u(\theta^b, t - \delta) \geq 0$ and $\frac{\partial}{\partial t} u(\theta^b, t + \delta) \leq 0$.

Let us start with the single crossing difference condition. Consider $\theta^b < \tilde{\theta}^b$ and $t < \tilde{t} \leq T$ and suppose that

$$u(\theta^b, t) \geq u(\theta^b, \tilde{t}). \tag{A.10}$$

We will show that $u(\tilde{\theta}^b, t) > u(\tilde{\theta}^b, \tilde{t})$. Define function

$$g(u|\theta^b, t) = e^{-ru}(v(\theta^b) - p^b_u)1\{u < t)\} + e^{-rt}(v(\theta^b) - p^s_t)1\{u \geq t\}.$$

Then

$$\int_0^T g(u|\theta^b, t) dF(\theta^s_u|\theta^b) \geq \int_0^T g(u|\theta^b, \tilde{t}) dF(\theta^s_u|\theta^b) \geq \int_0^T g(u|\theta^b, \tilde{t}) dF(\theta^s_u|\tilde{\theta}^b),$$

where the first inequality follows from $(A.10)$, the second inequality follows from the fact that $g(\cdot|\theta^b, \tilde{t})$ is decreasing and $F(\cdot|\tilde{\theta}^b)$ first-order stochastically dominates $F(\cdot|\theta^b)$ (as f is affiliated). This implies that

$$\begin{aligned} u(\theta^b, t) = \int_0^t & e^{-ru}(v(\theta^b) - p^b_u) dF(\theta^s_u|\theta^b) + (1 - F(\theta^s_t|\theta^b)) e^{-rt}(v(\theta^b) - p^s_t) \\ & \geq \int_0^{\tilde{t}} e^{-ru}(v(\theta^b) - p^b_u) dF(\theta^s_u|\tilde{\theta}^b) + (1 - F(\theta^s_{\tilde{t}}|\tilde{\theta}^b)) e^{-rt}(v(\theta^b) - p^s_t), \end{aligned}$$

or equivalently,

$$\begin{aligned} v(\theta^b) & \left(\int_0^t e^{-ru} dF(\theta^s_u|\theta^b) + (1 - F(\theta^s_t|\theta^b)) e^{-rt} - \int_0^{\tilde{t}} e^{-ru} dF(\theta^s_u|\tilde{\theta}^b) - (1 - F(\theta^s_{\tilde{t}}|\tilde{\theta}^b)) e^{-r\tilde{t}} \right) \\ & \geq p^s_t - \int_0^{\tilde{t}} e^{-ru} p^B_u dF(\theta^s_u|\tilde{\theta}^b) - (1 - F(\theta^s_{\tilde{t}}|\tilde{\theta}^b)) e^{-r\tilde{t}} p^s_{\tilde{t}}. \end{aligned} \tag{A.11}$$

We will show that the left-hand side of $(A.11)$ is positive and so, the left-hand side would increase if we substitute $v(\tilde{\theta}^b)$ instead of $v(\theta^b)$. This in turn implies that $u(\tilde{\theta}^b, t) > u(\tilde{\theta}^b, \tilde{t})$ and completes the proof of the strict single crossing difference. Let $h(u|t) = e^{-ru}1\{u < t)\} + e^{-rt}1\{u \geq t\}$.

Then the left-hand side of $(A.11)$ is equal to

$$
\begin{aligned}
&v(\theta^b)\left(\int_0^T h(u|t)dF(\theta_u^s|\theta^b) - \int_0^T h(u|\tilde{t})dF(\theta_u^s|\tilde{\theta}^b)\right) \\
\geq &v(\theta^b)\left(\int_0^T h(u|t)dF(\theta_u^s|\tilde{\theta}^b) - \int_0^T h(u|\tilde{t})dF(\theta_u^s|\tilde{\theta}^b)\right) \\
= &v(\theta^b)\int_0^T (h(u|t) - h(u|\tilde{t}))dF(\theta_u^s|\tilde{\theta}^b) > 0,
\end{aligned}
$$

where the first inequality follows from $F(\cdot|\tilde{\theta}^b)$ first-order stochastically dominates $F(\cdot|\theta^b)$ and $h(\cdot|t)$ decreasing, and the last term is strictly positive by $t < \tilde{t}$.

Now, let us show the second requirement of the SSCD condition. Suppose $\frac{\partial}{\partial t}u(\theta^b, t) = 0$. By taking the partial derivative

$$
e^{rt}\frac{\partial}{\partial t}u(\theta^b, t) = (p_t^s - p_t^b)f(\theta_t^s|\theta^b)\dot{\theta}_t^s - (1 - F(\theta_t^s|\theta^b))(r(v(\theta^b) - p_t^s) + \dot{p}_t^s),
$$

we get that

$$
\begin{aligned}
&e^{rt}\frac{\partial}{\partial t}u(\theta^b - \delta, t) = \\
&\qquad (p_t^s - p_t^b)f(\theta_t^s|\theta^b - \delta)\dot{\theta}_t^s - (1 - F(\theta_t^s|\theta^b - \delta))(r(v(\theta^b - \delta) - p_t^s) + \dot{p}_t^s) = \\
&\qquad (1 - F(\theta_t^s|\theta^b - \delta))\left((p_t^s - p_t^b)\frac{f(\theta_t^s|\theta^b - \delta)}{1 - F(\theta_t^s|\theta^b - \delta)}\dot{\theta}_t^s - (r(v(\theta^b - \delta) - p_t^s) + \dot{p}_t^s\right).
\end{aligned}
$$

Since $v(\theta^b - \delta) \leq v(\theta^b)$ and $\frac{f(\theta_t^s|\theta^b - \delta)}{1 - F(\theta_t^s|\theta^b - \delta)} \geq \frac{f(\theta_t^s|\theta^b)}{1 - F(\theta_t^s|\theta^b)}$ (by the affiliation of f), it follows that $\frac{\partial}{\partial t}u(\theta^b - \delta, t) \geq 0$. Showing that $\frac{\partial}{\partial t}u(\theta^b + \delta, t) \leq 0$ is analogous. *q.e.d.* □

Approximate the SBS with BNEs in $\mathcal{G}(\cdot|F^*)$ The next lemma constructs price-offer paths $p_\cdot^{s,\varepsilon}$ and $p_\cdot^{b,\varepsilon}$ and BNEs in $\mathcal{G}(p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon}|F^*)$ that approximate the SBS outcome.

Lemma 9. *For any $\varepsilon \geq 0$. Let*

$$
\begin{aligned}
p_t^{b,\varepsilon} &= \mathbf{p}(0) + (1-\alpha)\varepsilon + (1-\alpha)r(\xi+\varepsilon)t, \\
p_t^{s,\varepsilon} &= \mathbf{p}(1) - \alpha\varepsilon - \alpha r(\xi+\varepsilon)t, \\
\theta_t^{b,\varepsilon} &= v^{-1}\left(\mathbf{p}(1) + \alpha\xi - \alpha r(\xi+\varepsilon)t\right), \\
\theta_t^{s,\varepsilon} &= c^{-1}\left(\mathbf{p}(0) - (1-\alpha)\xi + (1-\alpha)r(\xi+\varepsilon)t\right);
\end{aligned}
\tag{A.12}
$$

for $t \leq T^\varepsilon = \frac{\mathbf{p}(1) - \mathbf{p}(0) - \varepsilon}{r(\xi+\varepsilon)}$. Then

1. *$(\theta_\cdot^{s,\varepsilon}, \theta_\cdot^{b,\varepsilon})$ constitutes a BNE in $\mathcal{G}(p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon}|F^*)$;*

2. *the outcome of* $(\theta_{\cdot}^{s,0},\theta_{\cdot}^{b,0})$ *in* $\mathcal{G}(p_{\cdot}^{s,0},p_{\cdot}^{b,0}|F^{*})$ *coincides with the SBS outcome;*

3. $\theta^{b}_{T^{\varepsilon}} > \theta^{s}_{T^{\varepsilon}} + \varepsilon/\bar{\ell}$;

4. *the outcome of* $(\theta_{\cdot}^{s,\varepsilon},\theta_{\cdot}^{b,\varepsilon},p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon})$ *converge uniformly to* $(\theta_{\cdot}^{s,0},\theta_{\cdot}^{b,0},p_{\cdot}^{s,0},p_{\cdot}^{b,0})$ *as* $\varepsilon \to 0$.

Proof. 1) $(\theta_{\cdot}^{s,\varepsilon},\theta_{\cdot}^{b,\varepsilon},p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon})$ satisfy the premise of Lemma 8 and so, $(\theta_{\cdot}^{s,\varepsilon},\theta_{\cdot}^{b,\varepsilon})$ constitutes a BNE in $\mathcal{G}(p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon}|F^{*})$.

2) We can verify using $(A.12)$ that $\theta^{s,0}_{T^0}=\theta^{b,0}_{T^0}$ and

$$\begin{aligned} p^{s,0}_t &= v(\theta^{b,0}_t)-\alpha\xi=\mathbf{p}(\theta^{b,0}_t),\\ p^{b,0}_t &= c(\theta^{s,0}_t)+(1-\alpha)\xi=\mathbf{p}(\theta^{s,0}_t). \end{aligned}$$

Since types $\theta^{s,0}_t$ and $\theta^{b,0}_t$ accept offers $p^{b,0}_t$ and $p^{s,0}_t$, resp., (3.1) obtains. Since threshold types $\theta^{s,0}_t$ and $\theta^{b,0}_t$ prefer to accept at time t rather than any other type $t \le T^0$, (3.2) and (3.3) obtain where $\theta^{*}=\theta^{s,0}_{T^0}=\theta^{b,0}_{T^0}$. Thus, the outcome of $(\theta_{\cdot}^{s,0},\theta_{\cdot}^{b,0})$ in $\mathcal{G}(p_{\cdot}^{s,0},p_{\cdot}^{b,0}|F^{*})$ coincides with the SBS outcome.

3) $v(\theta^{b,\varepsilon}_T)-c(\theta^{s,\varepsilon}_T)=\xi+\varepsilon$ and so, $v(\theta^{b,\varepsilon}_T)=v(\theta^{s,\varepsilon}_T)+\varepsilon \ge v(\theta^{b,\varepsilon}_T)-\bar{\ell}(\theta^{b,\varepsilon}_T-\theta^{s,\varepsilon}_T)+\varepsilon$ which implies $\theta^{b,\varepsilon}_T > \theta^{s,\varepsilon}_T+\varepsilon/\bar{\ell}$.

4) From $(A.12)$, $(\theta_{\cdot}^{s,\varepsilon},\theta_{\cdot}^{b,\varepsilon},p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon})$ converge pointwise to $(\theta_{\cdot}^{s,0},\theta_{\cdot}^{b,0},p_{\cdot}^{s,0},p_{\cdot}^{b,0})$ as $\varepsilon\to 0$ on a compact $[0,T^0]$, and by the continuity of $(\theta_{\cdot}^{s,\varepsilon},\theta_{\cdot}^{b,\varepsilon},p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon})$ and $(\theta_{\cdot}^{s,0},\theta_{\cdot}^{b,0},p_{\cdot}^{s,0},p_{\cdot}^{b,0})$, the convergence is also uniform. □

Approximate the BNEs in $\mathcal{G}(\cdot|F^{*})$ with BNEs in $\mathcal{G}(\cdot|F)$ For each $(\theta_{\cdot}^{s,\varepsilon},\theta_{\cdot}^{b,\varepsilon})$ in $\mathcal{G}(p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon}|F^{*})$, we construct an approximating sequence of BNEs $(\theta_{\cdot}^{s},\theta_{\cdot}^{b})$ in $\mathcal{G}(p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon}|F)$,$F \xrightarrow{p} F^{*}$.[38]

Lemma 10. *Let* $T=T^{\varepsilon}$,$\theta_{\cdot}^{s}=\theta_{\cdot}^{s,\varepsilon}$, $\theta_{\cdot}^{b}=\theta_{\cdot}^{b,\varepsilon}$ *and* p^s_t,p^b_t *be given by the differential equations* $(A.8)$ *with the terminal condition* $p^s_T=p^b_T$ *and the initial condition* $p^s_0=p^{s,\varepsilon}_0$. *Then* $(\theta_{\cdot}^{s},\theta_{\cdot}^{b})$ *constitute BNE in* $\mathcal{G}(p_{\cdot}^{s},p_{\cdot}^{b}|F)$ *and* $(\theta_{\cdot}^{s},\theta_{\cdot}^{b},p_{\cdot}^{s},p_{\cdot}^{b})$ *converge uniformly to* $(\theta_{\cdot}^{s,\varepsilon},\theta_{\cdot}^{b,\varepsilon},p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon})$ *as* $F \xrightarrow{p} F^{*}$.

Proof. To prove the convergence, we show that $p_{\cdot}^{s},p_{\cdot}^{b}$ converge pointwise to $p_{\cdot}^{s,\varepsilon},p_{\cdot}^{b,\varepsilon}$ as $F \xrightarrow{p} F^{*}$. Denote

$$\begin{aligned} \psi_1(t) &= \frac{(1-\alpha)r(\xi+\varepsilon)}{c'(\theta^{s,\varepsilon}_t)}\cdot\frac{f(\theta^{s,\varepsilon}_t|\theta^{b,\varepsilon}_t)}{1-F(\theta^{s,\varepsilon}_t|\theta^{b,\varepsilon}_t)},\\ \psi_2(t) &= \frac{\alpha r(\xi+\varepsilon)}{v'(\theta^{b,\varepsilon}_t)}\cdot\frac{f(\theta^{b,\varepsilon}_t|\theta^{s,\varepsilon}_t)}{F(\theta^{b,\varepsilon}_t|\theta^{s,\varepsilon}_t)}. \end{aligned}$$

[38] We do not explicitly index the sequence by corresponding F and ε and simply write $(\theta_{\cdot}^{s},\theta_{\cdot}^{b})$.

Using $(A.12)$ to compute $\dot{\theta}_t^{s,\varepsilon}$ and $\dot{\theta}_t^{b,\varepsilon}$, we rewrite the system $(A.8)$ as

$$\dot{p}_t^s = p_t^s(\psi_1(t) + r) - p_t^b\psi_1(t) - rv(\theta_t^{b,\varepsilon}),$$
$$\dot{p}_t^b = p_t^b(\psi_2(t) + r) - p_t^s\psi_2(t) - rc(\theta_t^{s,\varepsilon}),$$

By subtracting the second equation from the first and denoting $\Delta p_t = p_t^s - p_t^b$, we get

$$\Delta\dot{p}_t = \Delta p_t(\psi_1(t) + \psi_2(t) + r) - r(v(\theta_t^{b,\varepsilon}) - c(\theta_t^{s,\varepsilon}))$$

with the terminal condition $\Delta p_T = 0$, which we can solve to get

$$\Delta p_t = r\int_t^{T_\varepsilon}(v(\theta_u^{b,\varepsilon}) - c(\theta_u^{s,\varepsilon}))e^{-\int_t^u(\psi_1(s)+\psi_2(s)+r)ds}du.$$

We can now solve for individual price-offer paths. We have

$$\dot{p}_t^s = rp_t^s + \Delta p_t\psi_1(t) - rv(\theta_t^{b,\varepsilon}),$$

from which we get

$$\begin{aligned} p_t^s &= e^{rt}(\mathbf{p}(1) - \alpha\varepsilon) + \int_0^t(\Delta p_u\psi_1(u) - rv(\theta_u^{b,\varepsilon}))e^{r(t-u)}du, \\ p_t^b &= p_t^s - \Delta p_t. \end{aligned}$$

By Lemma 9, $\theta_t^b = \theta_t^{b,\varepsilon} > \theta_t^{s,\varepsilon} + \varepsilon/\bar{\ell} = \theta_t^s + \varepsilon/\bar{\ell}$ for all $t \le T$, and so, for F sufficiently far in the sequence, for all $t \le T$, $\frac{f(\theta_t^{s,\varepsilon}|\theta_t^{b,\varepsilon})}{1-F(\theta_t^{s,\varepsilon}|\theta_t^{b,\varepsilon})}$ and $\frac{f(\theta_t^{b,\varepsilon}|\theta_t^{s,\varepsilon})}{F(\theta_t^{b,\varepsilon}|\theta_t^{s,\varepsilon})}$ are bounded from above by $c_0\varepsilon$ for some constant c_0. This together with the fact that v' and c' are bounded from below by $\underline{\ell}$ implies that $|\psi_1(t)|$ and $|\psi_2(t)|$ converge to zero pointwise on $[0,T]$ as $F \xrightarrow{p} F^*$. Therefore, price-offer paths and their derivatives converge pointwise on $[0,T]$ as $F \xrightarrow{p} F^*$ and so, by the continuity of $p_\cdot^s, p_\cdot^b, p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon}$ and their derivatives on the compact $[0,T]$, the convergence is also uniform.

The derivatives of $p_\cdot^{s,\varepsilon}$ and $p_\cdot^{b,\varepsilon}$ are bounded away from zero (from above and below, resp.), and so for F sufficiently far in the sequence, $p_\cdot^s$ and $p_\cdot^b$ are strictly decreasing and increasing, resp. This together with the construction of $(\theta_\cdot^s, \theta_\cdot^b, p_\cdot^s, p_\cdot^b)$, implies that $(\theta_\cdot^s, \theta_\cdot^b, p_\cdot^s, p_\cdot^b)$ satisfies the conditions of Lemma 8 and so, $(\theta_\cdot^s, \theta_\cdot^b)$ constitutes a BNE in $\mathcal{G}(p_\cdot^s, p_\cdot^b|F)$. □

Approximate BNEs in $\mathcal{G}(\cdot|F)$ with PBEs in $\mathcal{G}(F,\Delta)$ We use grim trigger strategies to construct PBEs in $\mathcal{G}(F,\Delta)$ that approximate each BNE $(\theta_\cdot^s, \theta_\cdot^b)$ in $\mathcal{G}(p_\cdot^s, p_\cdot^b|F)$. On the equilibrium path, the seller makes decreasing offers q_n^s and the buyer makes increasing offers q_n^b. Offers do not depend on the type, but the acceptance of the opponent's offer does. Specifically, players follow threshold strategies on-path: in round n, all types of the seller below s_n accept q_{n-1}^b and types above s_n reject it and counter-offer q^s_n; all types of the buyer above b_n accept q^s_n and types below b_n reject it and counter-offer q^b_n. If there is a deviation from the equilibrium path, the play switches to the punishing path (described below) that punishes the deviator.

Construction of the Equilibrium Path: We first construct on-path strategies and show that no type wants to mimic another type in the acceptance decision. We construct the discrete-time approximation of $\theta^s_\cdot$ and $\theta^b_\cdot$ (defined in the previous step) using the Euler method: $s_{N+1} = 1$,$s_N = \theta^s_T$, $b_N = \theta^b_T$, and for $n < N = \lfloor \frac{T}{\Delta} \rfloor$, $s_n = s_{n+1} - \dot{\theta}^s_{(n+1)\Delta}\Delta$ and $b_n = b_{n+1} - \dot{\theta}^b_{(n+1)\Delta}\Delta$. We construct price-offer paths $q^s_\cdot$ and $q^b_\cdot$ backwards in time starting from N and $q^s_{N+1} = q^b_N = q^s_T$ as follows: for $n \leq N$,

$$v(b_n) - q^s_n = e^{-r\Delta_s} \frac{F(s_{n+1}|b_n) - F(s_n|b_n)}{1 - F(s_n|b_n)} (v(b_n) - q^b_n) + e^{-r\Delta} \frac{1 - F(s_{n+1}|b_n)}{1 - F(s_n|b_n)} (v(b_n) - q^s_{n+1}), \tag{A.13}$$

$$q^b_{n-1} - c(s_n) = e^{-r\Delta_b} \frac{F(b_n|s_n) - F(b_{n-1}|s_n)}{F(b_{n-1}|s_n)} (q^s_n - c(s_n)) + e^{-r\Delta} \frac{F(b_n|s_n)}{F(b_{n-1}|s_n)} (q^b_n - c(s_n)). \tag{A.14}$$

Denote by $(\bar{s}_\cdot, \bar{b}_\cdot, \bar{q}^s_\cdot, \bar{q}^b_\cdot)$ the linear extrapolation of $(s_\cdot, b_\cdot, q^s_\cdot, q^b_\cdot)$ to the continuos time on $[0, T]$.

Lemma 11. *$(\bar{s}_\cdot, \bar{b}_\cdot, \bar{q}^s_\cdot, \bar{q}^b_\cdot)$ converges uniformly to $(\theta^s_\cdot, \theta^b_\cdot, p^s_\cdot, p^b_\cdot)$ as $F \xrightarrow{p} F^*$. When both players use threshold strategies, no player wants to deviate to a different acceptance strategy.*

Proof. To prove the first part, the convergence of $\bar{s}_\cdot$ and $\bar{b}_\cdot$ is by construction. Next, rewrite equation $(A.13)$ as follows

$$\frac{1 - e^{-r\Delta}}{\Delta}(v(b_n) - q^s_n) - e^{-r\Delta}\frac{q^s_n - q^s_{n+1}}{\Delta} = \frac{F(s_{n+1}|b_n) - F(s_n|b_n)}{\Delta(1 - F(s_n|b_n))}(e^{-r\Delta_s}(v(b_n) - q^b_n) - e^{-r\Delta}(v(b_n) - q^s_{n+1})).$$

Since $\dot{\theta}^s_\cdot$ is positive and bounded uniformly on $[0, T)$, by the construction of s_n, there is an upper bound c_0 on $\frac{1}{\Delta}|s_{n+1} - s_n|$ for $n < N$. Choose F sufficiently close to F^* so that $\sup_{(\theta^s,\theta^b):|\theta_b - \theta_s| > \varepsilon/\bar{\ell}} f(\theta^s|\theta^b) < \varepsilon/\bar{\ell}$. Then since $s_n \leq s_N = \theta^s_T < \theta^b_T - \varepsilon/\bar{\ell} = b_N - \varepsilon/\bar{\ell} \leq b_n - \varepsilon/\bar{\ell}$, we have

$$0 \leq \frac{F(s_{n+1}|b_n) - F(s_n|b_n)}{\Delta(1 - F(s_n|b_n))} \leq \frac{|s_{n+1} - s_n|\varepsilon}{\Delta\bar{\ell}(1 - \varepsilon/\bar{\ell})} \leq \frac{c_0\varepsilon}{\bar{\ell}(1 - \varepsilon/\bar{\ell})} \equiv C\varepsilon,$$

for $n < N$. Thus,

$$\frac{1 - e^{-r\Delta}}{\Delta}(v(b_n) - q^s_n) - e^{-r\Delta}\frac{q^s_n - q^s_{n+1}}{\Delta} = C(\varepsilon) \tag{A.15}$$

where $C(\varepsilon)$ is some function that is bounded in absolute value by $C\varepsilon$. This implies that $\bar{q}^s_\cdot$ converges pointwise to $p^s_\cdot$ as $F \xrightarrow{p} F^*$, and also uniformly by the continuity of $\bar{q}^s_\cdot$ and $p^s_\cdot$ on a compact $[0, T]$. Moreover, the left-derivative of $\overline{q}^s_\cdot$ equals to $\frac{q^s_n - q^s_{n+1}}{\Delta}$ for some n, and from equation $(A.15)$, it converges uniformly to $\dot{p}^s_\cdot$ as $F \xrightarrow{p} F^*$. This implies that $\overline{q}^s_\cdot$ is strictly decreasing for F sufficiently far in the sequence. The uniform convergence of $\overline{q}^b_\cdot$ and its left-derivative is proven analogously.

To prove the second part, first observe that we can follows the same line of argument as the proof of the single crossing difference condition in Lemma 8 (Claim 1) to prove the following claim.

Claim 2. *Buyer's types $\theta^b > b_n$ and seller's types $\theta^s < s_n$, strictly prefer to accept in round $n-1$ to accepting in round n. Buyer's types $\theta^b < b_n$ and seller's types $\theta^s > s_n$, strictly prefer to accept in round $n+1$ to accepting in round n.*

Claim 2 implies that no player wants to deviate from the threshold strategy in the acceptance decision. □

Construction of the Punishing Path: We now construct punishing path for deviations to offers that are off the equilibrium path and show that such are not profitable. Suppose that the seller deviates from the price-offer path $q^s_\cdot$ in round n (the construction of the punishing path for the buyer is symmetric). Then specify that the buyer assigns probability 1 to the lowest remaining seller's type, s_{n-1}. The next lemma states that there is a continuation equilibrium that is efficient in deterring deviations to off-path offers.

Lemma 12 (Coasian Property). *Let $\underline{s} < \theta^s_T$ and $\overline{b} > \theta^b_T$. Suppose after some history, the buyer assigns probability 1 to type $\underline{s}$ and the seller's beliefs are $F(\theta^b|\theta^s, \theta^b < \overline{b})$. Then for any $\varepsilon_0 > 0$ there is $\overline{\Delta}$ (that does not depend on $\underline{s}$ and $\overline{b}$) such that for all $\Delta < \overline{\Delta}$, there is a continuation PBE strategies in which the seller's initial offer is below $\mathbf{p}(\underline{s}, 0) + \varepsilon_0$.*

Proof. We construct the continuation PBE in which the buyer's types pool on the unacceptable offer above $\overline{\mathbf{p}}(1)$ and the seller (with commonly known type $\underline{s}$) makes offers to screen the buyer's type. We can follow the step in the proof of Proposition 1 in Fudenberg et al. (1985) to construct the screening path of the seller in which the last seller's offer equals $\overline{\mathbf{p}}(\underline{s}, 0)$. By the Uniform Coase Conjecture in Ausubel and Deneckere (1989), there is $\overline{\Delta}$ (that does not depend on $\underline{s}$ and $\overline{b}$) such that for all $\Delta < \overline{\Delta}$, there is a PBE in which the seller's initial offer is below $\mathbf{p}(\underline{s}, 0) + \varepsilon$. To guarantee that the buyer does not have incentives to make acceptable offers, we specify that if the buyer deviates from making the unacceptable offer, the seller assigns probability 1 to type $\overline{b}$. Specify that after histories in which the seller assigns probability 1 to a certain type of the buyer (there are two possibilities: either 0 or $\overline{b}$), the continuation play is as in the complete information game.[39] Thus, if the buyer deviates, he trades at price at best $\underline{\mathbf{p}}(\underline{s}, \overline{b})$ which is

[39] This is the only place where we use the weakening of the support restriction on beliefs.

strictly higher than $\mathbf{p}(\underline{s}, 0) + \varepsilon_0$ when $\bar{b} > \theta_T^b > \underline{s}$ and ε is small. Therefore, such a deviation is not profitable for sufficiently small Δ. $\square$

We now show that players do not have incentives to deviate to off-path price offers. By Lemma 12, if such a type deviates in round n, she trades at a price at best $\mathbf{p}(s_n, 0) + \varepsilon_0$. Thus, the price of the punishing path converges uniformly to $\mathbf{p}(\theta_t^s)$ as $\Delta \to 0$ by Lemma 11. On the other hand, on the equilibrium path the buyer's offers p_n^b converge uniformly to $p_t^s = \mathbf{p}(\theta_t^s) + (1-\alpha)\varepsilon > \mathbf{p}(\theta_t^s)$ by Lemmas 9 and 10. Since $c(1) - c(0) < \min\{\frac{1-\alpha}{\alpha}, \frac{\alpha}{1-\alpha}\}\xi$,

$$\begin{aligned} p_t^s \geq p_T^s &= \mathbf{p}(1) - \alpha\varepsilon - \alpha r(\xi + \varepsilon)T_\varepsilon \\ &= \mathbf{p}(1) - \alpha(\mathbf{p}(1) - \mathbf{p}(0)) \\ &= c(1) + (1-\alpha)\xi - \alpha(c(1) - c(0)) > c(1). \end{aligned}$$

Therefore, any deviation from the on-path offers is not profitable when Δ is sufficiently small. This completes the construction of the PBEs in $\mathcal{G}(F, \Delta)$ and completes the proof of part 2 of Theorem 1.

A.2 Proofs for the OTC Model

Equilibrium Characterization

Step 1: Proof of Lemma 1. We first derive the steady-state distribution of times spent in the match. For $\theta \in \Theta_L \cup \Theta_M$ and $u \in [0, t(\theta)]$, let $G(\theta, u)$ be the mass of sellers that have spent time u negotiating the price of an asset of quality θ. During the time interval du, a fraction $(y_u + y_d)du$ of matches is destroyed due to the switching of intrinsic types, and for an asset of quality θ, a mass $\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)du$ of investors enters the bargaining stage. Hence, the change in the mass of sellers that have spent in the match less than u is $(1 - y_u du - y_d du)G(\theta, u - du) + \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)du - G(\theta, u)$, which equals 0 in the steady-state. Thus,

$$\tfrac{\partial}{\partial u} G(\theta, u) = -(y_u + y_d)G(\theta, u) + \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta), \tag{A.16}$$

which together with $G(\theta, 0) = 0$ gives $G(\theta, u) = \frac{1 - e^{-(y_u + y_d)u}}{y_u + y_d}\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)$. The total mass of sellers in the bargaining stage for asset θ is equal to $\mu_{sm}(\theta)$ which translates into $G(\theta, t(\theta)) = \mu_{sm}(\theta)$ or equivalently

$$\mu_{sm}(\theta) = \frac{1 - e^{-(y_u + y_d)t(\theta)}}{y_u + y_d}\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta). \tag{A.17}$$

Let $w(\theta)$ be the intensity with which investors leave the match. Note that this is also the instantaneous trade volume for the asset of quality θ. During the time interval du, sellers that have already spent time $[t(\theta)-du, t(\theta)]$ in the bargaining stage complete their trades. Thus, $w(\theta)=\frac{\partial}{\partial u}G(\theta,t(\theta))$ or

$$w(\theta)=\lambda M_{bu}(\phi)\mu_{su}(\theta)e^{-(y_u+y_d)t(\theta)}\sigma(\theta). \tag{A.18}$$

Now, we derive the distribution M. For $\theta\in\Theta_I$, $\mu_{su}(\theta)=\frac{y_d}{y_u+y_d}, \mu_{bu}(\theta)=\frac{y_u}{y_u+y_d}, \mu_{sm}(\theta)=\mu_{bm}(\theta)=0$ and we only consider $\theta\in\Theta_L\cup\Theta_M$. In the steady state, $\mu_{su}(\theta), \mu_{bu}(\theta), M_{bu}(\phi), M_{su}(\phi)$ stay constant over time and so,

$$\begin{cases} y_d\mu_{sm}(\theta)+y_d\mu_{bu}(\theta)=y_u\mu_{su}(\theta)+\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta),\\ y_u\mu_{sm}(\theta)+y_u\mu_{su}(\theta)+w(\theta)=y_d\mu_{bu}(\theta),\\ y_uM_{bm}(\Theta_L\cup\Theta_M)+y_uM_{su}(\phi)=y_dM_{bu}(\phi)+\lambda M_{bu}(\phi)\left(\int_0^1\mu_{su}(\theta)\sigma(\theta)d\theta\right),\\ y_dM_{bm}(\Theta_L\cup\Theta_M)+y_dM_{bu}(\phi)+\int_0^1 w(\theta)d\theta=y_uM_{su}(\phi),\end{cases} \tag{A.19}$$

where the left-hand sides are the inflows into and the right-hand sides are the outflows from $\mu_{su}(\theta), \mu_{bu}(\theta), M_{bu}(\phi), M_{su}(\phi)$, respectively. Combining the system $(A.19)$ with the balance conditions $(5.2), (5.1), (5.3)$, and $(A.17)-(A.18)$, we get:

$$\begin{cases} y_d\mu_{sm}(\theta)+y_d\mu_{bu}(\theta)-y_u\mu_{su}(\theta)-\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)=0,\\ y_u\mu_{sm}(\theta)+y_u\mu_{su}(\theta)-y_d\mu_{bu}(\theta)+\lambda M_{bu}(\phi)\mu_{su}(\theta)e^{-(y_u+y_d)t(\theta)}\sigma(\theta)=0,\\ \mu_{su}(\theta)+\mu_{bu}(\theta)+\mu_{sm}(\theta)=1,\\ y_uM_{sm}(\Theta_L\cup\Theta_M)+y_uM_{su}(\phi)-y_dM_{bu}(\phi)-\lambda M_{bu}(\phi)\left(\int_0^1\mu_{su}(\theta)\sigma(\theta)d\theta\right)=0,\\ M_{su}(\phi)+M_{bu}(\phi)+M_{sm}(\Theta_L\cup\Theta_M)=a-1,\\ (y_u+y_d)\mu_{sm}(\theta)-(1-e^{-(y_u+y_d)t(\theta)})\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)=0,\\ y_dM_{sm}(\Theta_L\cup\Theta_M)+y_dM_{bu}(\phi)-y_uM_{su}(\phi)+\lambda M_{bu}(\phi)\int_0^1\mu_{su}(\theta)e^{-(y_u+y_d)t(\theta)}\sigma(\theta)d\theta=0.\end{cases} \tag{A.20}$$

The rank of the system is five and we eliminate the last two equations to guarantee the full rank. Using $\Lambda_s=\lambda M_{bu}(\phi)$ we get from the first three equations in $(A.20)$

$$\begin{cases} \mu_{su}(\theta) &= \dfrac{y_d}{y_u+y_d+\Lambda_s\sigma(\theta)},\\ \mu_{bm}(\theta) &= \dfrac{\Lambda_s\sigma(\theta)(1-e^{-(y_u+y_d)t(\theta)})y_d}{(y_u+y_d)(y_u+y_d+\Lambda_s\sigma(\theta))},\\ \mu_{bu}(\theta) &= \dfrac{y_u(y_u+y_d)+\Lambda_s\sigma(\theta)(y_u+y_de^{-(y_u+y_d)t(\theta)})}{(y_u+y_d)(y_u+y_d+\Lambda_s\sigma(\theta))}.\end{cases} \tag{A.21}$$

In system $(A.20)$, subtracting the forth equation from the fifth multiplied by y_u and using

$\Lambda_s = \lambda M_{bu}(\phi)$, we get

$$(y_u + y_d)\frac{\Lambda_s}{\lambda} = (a-1)y_u - \Lambda_s \left(\int_0^1 \mu_{su}(\theta)\sigma(\theta)d\theta \right),$$

which after plugging in $\mu_{su}(\theta)$ from $(A.21)$ gives the equation (5.4). The left-hand side of (5.4) is strictly increasing in Λ_s and converges to infinity as $\Lambda_s \to \infty$ and the right-hand side is strictly decreasing in Λ_s unless $\sigma(\theta) = 0$ for almost all θ which does not hold in equilibrium. At $\Lambda_s = 0$, the left-hand side is zero and the right-hand side equals $\frac{y_u}{y_u+y_d}(a-1) > 0$. Thus, equation (5.4) has a unique positive solution.

Let us summarize. Quantities $\mu_{su}(\theta), \mu_{bu}(\theta), \mu_{bm}(\theta), \mu_{sm}(\theta)$ are given by $(A.21)$, $M_{bu}(\phi) = \frac{\Lambda_s}{\lambda}$ and $M_{su}(\phi)$ can be found from the fifth equation in $(A.20)$. Thus, the distribution M is characterized by $(\sigma(\cdot), x(\cdot), \Lambda_s)$. Using the expression for $\mu_{su}(\theta)$ in the first line of $(A.21)$, we find that Λ_b is given by (5.5). $F_L(\theta) = \frac{\int_{[0,\theta]\cap\Theta_L} \mu_{su}(\theta)d\theta}{M_{su}(\Theta_L)} = \frac{|[0,\theta]\cap\Theta_L|}{L}$ and so, F_L is uniform on $\theta \in \Theta_L$.

If $\sigma(\cdot)$ increases weakly pointwise, then the right-hand side of (5.4) decreases and so, it is necessary for equation (5.4) to hold that Λ_s decreases which by the equation (5.5) and the fact that L (weakly) increases when $\sigma(\cdot)$ (weakly) increases leads to an increase in Λ_b. $\square$

Step 2: At this step, we want to express value functions for every θ through $\Lambda_s, \Lambda_b, \sigma(\theta)$ and $x(\theta)$. Denote by $U_s(\theta)$ the utility of the seller who owns an asset of quality θ and does not participate in the search market. Setting $\sigma(\theta) = 0$ in equations (5.8) and (5.10), we get

$$U_s(\theta) = \frac{1}{r}\left(d + k\theta - \frac{r+y_d}{r+y_u+y_d} \right). \tag{A.22}$$

For $\theta \in \Theta_I$, $V_{su}(\theta) = U_s(\theta)$. The next lemma simplifies equations (5.7), (5.8), (5.9), (5.10) and expresses $V_{bu}(\theta)$ and $V_{su}(\phi)$ through $V_{bu}(\phi)$ and $V_{su}(\theta)$ (the proof is straightforward and omitted).

Lemma 13. *For all $\theta \in [0,1]$,*

$$V_{bu}(\theta) = \frac{d + k\theta + y_d V_{su}(\theta)}{r + y_d}, \tag{A.23}$$

$$V_{su}(\phi) = \frac{y_u V_{bu}(\phi)}{r + y_u}, \tag{A.24}$$

$$V_{bu}(\phi) = \Lambda_b \frac{r+y_u}{r\rho} \left(\mathbb{E}\left[V_{bm}(\theta) | \theta \in \Theta_L \right] - V_{bu}(\phi) \right), \tag{A.25}$$

$$V_{su}(\theta) = U_s(\theta) + \sigma(\theta)\Lambda_s \frac{r+y_d}{r\rho} \left(V_{sm}(\theta) - V_{su}(\theta) \right). \tag{A.26}$$

Recall that we define the status quo $(\hat{v}, \hat{c}(\theta))$ as the outcome that gives investors the stream payoffs they receive during the negotiation process. Thus, $\hat{c}(\theta)$ is given by the Bellman equation

$$r\hat{c}(\theta) = d + k\theta - 1 + y_u(V_{bu}(\theta) - \hat{c}(\theta)) + y_d(V_{su}(\theta) - \hat{c}(\theta)),$$

and so, using $(A.22)$ and $(A.23)$, we get

$$\hat{c}(\theta) = \frac{1}{\rho}(d + k\theta - 1 + y_u V_{bu}(\theta) + y_d V_{su}(\theta)) = \frac{r}{r+y_d}U_s(\theta) + \frac{y_d}{r+y_d}V_{su}(\theta).$$

Analogously, $\hat{v}$ is given by the Bellman equation

$$r\hat{v} = y_u(V_{bu}(\phi) - \hat{v}) + y_d(V_{su}(\phi) - \hat{v}),$$

and so, using $(A.24)$,

$$\hat{v} = \frac{1}{\rho}(y_u V_{bu}(\phi) + y_d V_{su}(\phi)) = \frac{y_u}{r+y_u}V_{bu}(\phi).$$

Functions v and c introduced in (3.4) and (3.5) are given by

$$c(\theta) = \frac{r}{r+y_d}U_s(\theta) + \frac{y_d}{r+y_d}V_{su}(\theta) - \frac{y_u}{r+y_u}V_{bu}(\phi), \tag{A.27}$$

$$v(\theta) = \frac{d+k\theta}{r+y_d} + \frac{y_d}{r+y_d}V_{su}(\theta) - \frac{y_u}{r+y_u}V_{bu}(\phi). \tag{A.28}$$

Observe that $v(\theta) - c(\theta) = \frac{1}{\rho} \equiv \xi$. Thus, the price of trade is given by

$$\begin{aligned} p(\theta) &= (1-\alpha)\frac{d+k\theta}{r+y_d} + \alpha\frac{r}{r+y_d}U_s(\theta) + \frac{y_d}{r+y_d}V_{su}(\theta) - \frac{y_u}{r+y_u}V_{bu}(\phi) \\ &= \frac{d+k\theta}{r+y_d} + \frac{y_d}{r+y_d}V_{su}(\theta) - \frac{y_u}{r+y_u}V_{bu}(\phi) - \alpha\xi. \end{aligned} \tag{A.29}$$

The next lemma expresses the value functions of matched investors through $x(\theta)$, $V_{su}(\theta)$ and $V_{bu}(\phi)$.

Lemma 14. *For any $\theta \in [0,1]$,*

$$V_{bm}(\theta) = \alpha\xi x(\theta) + \frac{y_u}{r+y_u}V_{bu}(\phi), \tag{A.30}$$

$$V_{sm}(\theta) = (1-\alpha)\xi x(\theta) + \frac{r}{r+y_d}U_s(\theta) + \frac{y_d}{r+y_d}V_{su}(\theta). \tag{A.31}$$

Proof. We solve the differential equation (5.11) with the terminal condition $V_{bm}(t(\theta),\theta) = V_{bu}(\theta) - p(\theta)$ to get

$$V_{bm}(t,\theta) = (V_{bu}(\theta) - p(\theta))\, e^{-\rho(t(\theta)-t)} + \frac{y_u V_{bu}(\phi)}{r+y_u}\left(1 - e^{-\rho(t(\theta)-t)}\right).$$

Using $(A.23), (A.29)$, and $V_{bm}(0,\theta) = V_{bm}(\theta)$, we get $(A.30)$. The derivation of $(A.31)$ is symmetric. $\square$

Proof of Lemma 2. Combining $(A.23)$ and $(A.30)$, we get (5.14). The buyer prefers to trade with the seller of asset θ if and only if $V_{bm}(\theta) \geq V_{bu}(\phi)$, or combining $(A.30)$ and (5.14), we get the condition (5.13). $\square$

To complete the derivation of value functions, we need to find $V_{su}(\theta)$. It follows from $(A.26)$ and $(A.31)$ that for $\theta \in \Theta_L \cup \Theta_M$ function V_{su} is given by

$$V_{su}(\theta) = U_s(\theta) + (1-\alpha)\frac{r + y_d}{r}\frac{\sigma(\theta)\Lambda_s}{\rho + \sigma(\theta)\Lambda_s}\xi x(\theta) \tag{A.32}$$

Equation $(A.32)$ implies that $V_{su}(\theta) > U_s(\theta)$ whenever $x(\theta) > 0$ and so, sellers always prefer to trade.

Step 3: We will derive the liquidity profile x for given $v(\cdot)$ and $c(\cdot)$. Consider asset qualities $\theta \in (\theta^*, 1]$ for which the bargaining delay is pinned down by (3.2). We can rewrite (3.2) as

$$\theta \in \arg\max_{\theta' \in [\theta^*, 1]} x(\theta')(v(\theta) - p(\theta')). \tag{A.33}$$

By the single-crossing condition, $x(\cdot)$ is weakly increasing on $(\theta^*, 1]$.

By the envelope theorem (Milgrom and Segal (2002)), function $x(\theta)(v(\theta) - p(\theta))$ is absolutely continuous and so, since $v(\cdot)$ and $p(\cdot)$ are continuous on $\Theta_L \cup \Theta_M$, $x(\cdot)$ is absolutely continuous on $(\theta^*, 1] \cap (\Theta_L \cup \Theta_M)$. Thus, at the differentiability points of $x(\cdot)$ the first-order condition for $(A.33)$ is

$$x'(\theta)(v(\theta) - p(\theta)) - x(\theta)p'(\theta) = 0, \tag{A.34}$$

which we can rewrite as

$$\frac{x'(\theta)}{x(\theta)} = \frac{v'(\theta)}{\alpha\xi}. \tag{A.35}$$

Since $v'(\cdot) \geq 0$ almost everywhere, $x'(\theta) \geq 0$ almost everywhere on $(\theta^*, 1] \cap (\Theta_L \cup \Theta_M)$.

By the analogous argument for $\theta < \theta^*$, we have that $x(\cdot)$ is weakly decreasing on $[0, \theta^*)$ (thus, $x(\cdot)$ has a U shape on $[0, 1]$), and for $[0, \theta^*) \cap (\Theta_L \cup \Theta_M)$, $x(\theta)$ satisfies

$$\frac{x'(\theta)}{x(\theta)} = -\frac{c'(\theta)}{(1-\alpha)\xi}. \tag{A.36}$$

The following lemma follows immediately from $(A.35)$ and $(A.36)$.

Lemma 15. *For differentiability points $\theta \in (\theta^*, 1] \cap (\Theta_L \cup \Theta_M)$ of $v(\cdot)$, $x'(\theta) = 0$ if and only if $v'(\theta) = 0$, and for differentiability points $\theta \in [0, \theta^*) \cap (\Theta_L \cup \Theta_M)$ of $c(\cdot)$, $x'(\theta) = 0$ if and only if $c'(\theta) = 0$.*

Proof of Lemma 3. The analysis proceeds in a series of claims.

Claim 3. *If $x(\theta) = \underline{x}$ on an interval (θ', θ''), then $\sigma(\theta) \in (0,1)$ for almost every $\theta \in (\theta', \theta'')$.*

Proof. Suppose that $x(\theta) = \underline{x}$, but $\sigma(\theta) = 0$ for $\theta \in (\theta', \theta'')$ (the argument is identical for $\sigma(\theta) = 1$). By $(A.32)$, V_{su} is strictly increasing on (θ', θ'') and so, by $(A.27)$ and $(A.28)$, v and c are strictly increasing, which contradicts Lemma 15. *q.e.d.*

Claim 4. *There exist $\check{\theta} \leq \underline{\theta} \leq \theta^* \leq \overline{\theta} \leq \check{\theta}$ such that $\Theta_L = [0, \check{\theta}] \cup [\hat{\theta}, 1]$, $\Theta_M = (\check{\theta}, \underline{\theta}] \cup [\overline{\theta}, \hat{\theta})$,* and $\Theta_I = (\underline{\theta}, \overline{\theta})$.

Proof. By Lemma 2, buyers accept only asset qualities with $x(\theta) \geq \underline{x}$. Since $x(\cdot)$ has a U shape, there exist $\check{\theta} \leq \underline{\theta} \leq \theta^* \leq \overline{\theta} \leq \check{\theta}$ such that $x(\theta) \geq \underline{x}$ on $[0, \underline{\theta}] \cup [\overline{\theta}, 1]$ and $x(\theta) > \underline{x}$ on $[0, \check{\theta}] \cup [\hat{\theta}, 1]$, which combined with Claim 3 gives the result. *q.e.d.*

Claim 5. $\overline{\theta} = \hat{\theta}$.

Proof. Suppose to contradiction $\overline{\theta} < \hat{\theta}$. Then there exist a decreasing sequence $(\theta_i')_{i=1}^{\infty} \subset [\hat{\theta}, 1]$ and an increasing sequence $(\theta_i'')_{i=1}^{\infty} \subset [\overline{\theta}, \hat{\theta})$ both converging to $\hat{\theta}$. Lemma 15 implies that $v(\theta_i'')$ is constant for all i, and so from $(A.28)$ and $(A.32)$, $\sigma(\theta_i'')$ is decreasing in i. On the other hand, $\sigma(\theta_i') = 1$ for all i. Since $x(\hat{\theta}) = \underline{x}$, we get the contradiction to the continuity of $v(\cdot)$ at $\hat{\theta}$. *q.e.d.*

Claim 6. *$\check{\theta} < \hat{\theta}$ implies $\check{\theta} < \underline{\theta}$.*

Proof. Suppose to contradiction that $\check{\theta} = \underline{\theta} < \hat{\theta}$. Then there exist an increasing sequence of $(\theta_i')_{i=1}^{\infty} \subset [0, \check{\theta}]$ and a decreasing sequence $(\theta_i'')_{i=1}^{\infty} \subset \Theta_I$ both converging to $\underline{\theta}$. We have $x(\theta_i'') = 0$ and $x(\theta_i') > \underline{x} > 0$. From $(A.27)$ and $(A.32)$, this implies that $c(\theta_i') > c(\theta_i'')$, which contradicts the monotonicity of $c(\cdot)$ as $\theta_i' < \theta_i''$. *q.e.d.*

It follows from Claims 1-4 that the only possibilities are: a) $\check{\theta} = \underline{\theta} = \overline{\theta} = \check{\theta}$, b) $\check{\theta} < \underline{\theta} = \overline{\theta} = \hat{\theta}$, c) $\check{\theta} < \underline{\theta} < \overline{\theta} = \hat{\theta}$.

Finally, we obtain equation (5.15) by solving $(A.35)$ and $(A.36)$ with the initial condition $x(1) = x(0) = 1$ (recall, by the definition of the SBS $t(1) = t(0) = 0$). □

Combining the steps: We now combine all the steps to reduce the problem of finding equilibria to the problem of finding Λ_s and L.

Lemma 16.

$$\theta = 1 + \frac{r}{k}\alpha\xi \ln x(\theta) + \frac{y_d}{k}(1-\alpha)\xi\frac{\Lambda_s}{\rho + \Lambda_s}(1 - x(\theta)), \text{ for } \theta \geq \hat{\theta}, \tag{A.37}$$

$$\theta = -\frac{r}{k}(1-\alpha)\xi \ln x(\theta) + \frac{y_d}{k}(1-\alpha)\xi\frac{\Lambda_s}{\rho + \Lambda_s}(1 - x(\theta)), \text{ for } \theta \leq \check{\theta}. \tag{A.38}$$

Moreover, $v(\cdot)$ and $c(\cdot)$ are strictly increasing on Θ_L.

Proof. For almost every $\theta > \hat{\theta}$, plugging $v'(\theta)$ from $(A.28)$ into $(A.35)$, we get

$$\frac{x'(\theta)}{x(\theta)} = \frac{k + y_d V'_{su}(\theta)}{\alpha\xi(r + y_d)}. \tag{A.39}$$

By $(A.32)$,

$$V'_{su}(\theta) = \frac{k}{r} + (1-\alpha)\frac{r + y_d}{r}\frac{\Lambda_s}{\rho + \Lambda_s}\xi x'(\theta). \tag{A.40}$$

Since $x'(\theta) \geq 0$ for $\theta > \hat{\theta}$, it follows from $(A.28)$ and $(A.40)$ that $v'(\theta) > 0$. Further, combining $(A.39)$ and $(A.40)$, we get

$$x'(\theta) = \frac{k}{\xi r\left(\frac{\alpha}{x(\theta)} - \frac{y_d}{r}\frac{\Lambda_s}{\rho+\Lambda_s}(1-\alpha)\right)}, \tag{A.41}$$

which together with $x(1) = 1$ gives $(A.37)$.

Analogously, plugging in $c'(\theta)$ from $(A.27)$ into $(A.36)$,

$$\frac{x'(\theta)}{x(\theta)} = -\frac{rU'_s(\theta) + y_d V'_{su}(\theta)}{(1-\alpha)\xi(r + y_d)},$$

and using $(A.40)$ for $V'_{su}(\theta)$, we get that $c'(\cdot)$ is strictly decreasing for $\theta < \check{\theta}$ and that

$$x'(\theta) = -\frac{k}{\xi r(1-\alpha)\left(\frac{1}{x(\theta)} + \frac{y_d}{r}\frac{\Lambda_s}{\rho+\Lambda_s}\right)}, \tag{A.42}$$

which together with $x(0) = 1$ gives $(A.38)$. $\square$

We next express thresholds $\hat{\theta}, \check{\theta}, \underline{\theta}$, and L through $\underline{x}$. By Lemma 16 and the fact that $\underline{x} = x(\check{\theta}) = x(\hat{\theta})$:

$$\hat{\theta} = 1 + \frac{r}{k}\alpha\xi \ln \underline{x} + \frac{y_d}{k}(1-\alpha)\xi\frac{\Lambda_s}{\rho+\Lambda_s}(1-\underline{x}), \tag{A.43}$$

$$\check{\theta} = -\frac{r}{k}(1-\alpha)\xi \ln \underline{x} + \frac{y_d}{k}(1-\alpha)\xi\frac{\Lambda_s}{\rho+\Lambda_s}(1-\underline{x}). \tag{A.44}$$

It follows from $(A.43)$ and $(A.44)$ that

$$L = 1 - \hat{\theta} + \check{\theta} = -\frac{r\xi}{k}\ln \underline{x}. \tag{A.45}$$

For each $\theta \in \Theta_M$, $x(\theta) = \underline{x}$ and so, $c(\theta) = c(\check{\theta})$ by Lemma 15. Therefore, by $(A.27)$,

$$V_{su}(\theta) = V_{su}(\check{\theta}) - \frac{r}{y_d}(U_s(\theta) - U_s(\check{\theta})). \tag{A.46}$$

Using $(A.32)$ and $x(\check{\theta}) = \underline{x}$,

$$V_{su}(\theta) - U_s(\theta) = \frac{r + y_d}{r}\left(\frac{k}{y_d}(\check{\theta} - \theta) + (1-\alpha)\frac{\Lambda_s}{\rho + \Lambda_s}\xi\underline{x}\right). \tag{A.47}$$

Threshold $\underline{\theta}$ is determined as the minimum of $\hat{\theta}$ and the solution to the equation $U_s(\underline{\theta}) = V_{su}(\underline{\theta})$ and so, from $(A.47)$,

$$\underline{\theta} = \min\left\{\hat{\theta}, \check{\theta} + (1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho + \Lambda_s}\xi\underline{x}\right\}. \tag{A.48}$$

This completes the description of $x(\cdot)$ for given Λ_s and $\underline{x}$. The next lemma determines $\sigma(\cdot)$.

Lemma 17. *For given Λ_s and $\underline{x}$,*

$$\sigma(\theta) = \begin{cases} 1, & \text{if } \theta \in [0, \check{\theta}] \cup [\hat{\theta}, 1], \\ 0, & \text{if } \theta \in [\underline{\theta}, \hat{\theta}), \\ \frac{\rho}{\Lambda_s}\frac{(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x} - \frac{k}{y_d}(\theta - \check{\theta})}{(1-\alpha)\frac{\rho}{\rho+\Lambda_s}\xi\underline{x} + \frac{k}{y_d}(\theta - \check{\theta})}, & \text{if } \theta \in (\check{\theta}, \underline{\theta}). \end{cases} \tag{A.49}$$

Proof. We only need to determine $\sigma(\theta)$ for $\theta \in \Theta_M$. It follows from $(A.32), (A.47)$ and $x(\theta) = \underline{x}$:

$$\sigma(\theta) = \frac{\rho}{\Lambda_s}\frac{\frac{r}{r+y}(V_{su}(\theta) - U_s(\theta))}{(1-\alpha)\xi\underline{x} - \frac{r}{r+y}(V_{su}(\theta) - U_s(\theta))} = \frac{\rho}{\Lambda_s}\left(\frac{(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x} - \frac{k}{y_d}(\theta - \check{\theta})}{(1-\alpha)\frac{\rho}{\rho+\Lambda_s}\xi\underline{x} + \frac{k}{y_d}(\theta - \check{\theta})}\right).$$

□

Proof of Theorem 2. For given Λ_s and $\underline{x}$, we can determine equilibrium strategy $\sigma(\cdot)$ from Lemma 17 and $x(\cdot)$ from Lemma 16 where $\check{\theta}, \underline{\theta}$, and $\hat{\theta}$ are expressed through Λ_s and $\underline{x}$ from $(A.43), (A.44)$, and $(A.48)$. Lemma 1 describes the steady-state distribution for given $\sigma(\cdot)$ and $x(\cdot)$. Moreover, there is a one-to-one mapping between$\underline{x}$ and L given by $(A.45)$ whenever $L < 1$. Thus, equilibrium is pinned down by Λ_s and L. We next characterize these quantities. Lemma 21 in the Online Appendix shows that equation (5.4) implies equation (5.17). Next, we derive (5.16). Combining (5.5) and (5.13), we get

$$\rho \geq \frac{\lambda y_d}{y_u + y_d + \Lambda_s}\int_{x(\theta) > \underline{x}}\left(\frac{x(\theta)}{\underline{x}} - 1\right)d\theta, \tag{A.50}$$

which holds as equality whenever $L < 1$. Given $(A.37)$ and $(A.38)$, we can explicitly calculate

$$X = \int_{x(\theta) > \underline{x}} x(\theta)d\theta = \int_{\hat{\theta}}^{1} x(\theta)d\theta + \int_{0}^{\check{\theta}} x(\theta)d\theta = \int_{\underline{x}}^{1} x\frac{d\theta(x)}{dx}dx - \int_{\underline{x}}^{1} x\frac{d\theta(x)}{dx}dx = \frac{r\xi}{k}(1 - \underline{x}) \tag{A.51}$$

and combined with $(A.45)$ and $(A.50)$ this gives (5.16). Therefore, Λ_s and L are determined by the system (5.16) and (5.17). Let function $\Lambda_s^1(L)$ and $\Lambda_s^2(L)$ be implicit functions given by equations (5.16) and (5.17), resp. At $L = 1$, any Λ_s above $\frac{\lambda y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}} - 1\right) - 1\right) - y_u - y_d$ satisfies (5.16). Moreover, $\Lambda_s^2(L) > 0$ for any L (to see this, note that there is no $L \in [0, 1]$ that solves (5.17) with $\Lambda_s = 0$), while $\Lambda_s^1(0) = -y_u - y_d < 0$. Therefore, the solution to (5.16) and (5.17) always exists. In the Online Appendix, Lemma 23 shows that Λ_s^1 is strictly increasing and Λ_s^2 is strictly decreasing and so, the uniqueness obtains.

Condition $\alpha \geq \frac{y_d}{r+y_d}$ guarantees that functions v and c are indeed monotone (as conjectured). From $(A.42)$, $x'(\theta) < 0$ for $\theta < \check{\theta}$ and so $(A.36)$ implies $c'(\theta) < 0$ for $\theta < \check{\theta}$. Whenever $\alpha \geq \frac{y_d}{r+y_d}$, $\frac{\alpha}{x(\theta)} \geq \alpha \geq \frac{y_d}{r}(1 - \alpha) > \frac{y_d}{r}\frac{\Lambda_s}{\rho+\Lambda_s}(1 - \alpha)$ and so from $(A.41)$, $x'(\theta) > 0$ for $\theta \geq \hat{\theta}$. Hence, it follows from $(A.36)$ that $v'(\theta) > 0$ for $\theta > \hat{\theta}$. By Lemma 15, $c(\theta)$ is constant on $[\check{\theta}, \underline{\theta}]$. On $[\underline{\theta}, \hat{\theta})$, $c(\theta) = U_s(\theta) - \frac{y_u}{r+y_u}V_{bu}(\phi)$. Finally, $\lim_{\theta\uparrow\hat{\theta}} v(\theta) < \lim_{\theta\downarrow\hat{\theta}} v(\theta)$, as qualities just below $\hat{\theta}$ are not traded, while qualities just above $\hat{\theta}$ are accepted. Therefore, since $v'(\theta) = c'(\theta)$, v and c are increasing on $[0, 1]$.

Finally, note that d does not enter in the expressions characterizing $M, x(\cdot), \sigma(\cdot)$. □

Difference between Bargaining and Search Delays

Proof of Proposition 2. 1) As $k \to 0$, the right-hand side of (5.16) converges to $-y_u - y_d$, thus, $L^\star = 1$. It follows from equation $(A.45)$, than $\underline{x}^\star = 1$. Equation (6.1) is obtained from (5.17) by setting $L = 1$. Suppose $\Lambda_s^\star \to \infty$ as $\lambda \to \infty$. From (6.1), $\lim_{\lambda\to\infty}\frac{\Lambda_s^\star}{\lambda} = \frac{y_u}{y_u+y_d}a - 1$ and so, $1 < \frac{y_u}{y_u+y_d}a$. Now, suppose $\Lambda_s^\star \to C \in (0, \infty)$ as $\lambda \to \infty$. From (6.1), $\frac{y_u}{y_u+y_d}(a - 1) = \frac{y_d}{y_u+y_d}\frac{C}{y_u+y_d+C}$ and so, $1 > \frac{y_u}{y_u+y_d}a$.

2a) First observe that if $\Lambda_s \to \Lambda_s^{\star\star} < \infty$, then $L^{\star\star} = 0$ by (5.16). Thus, whenever $L^{\star\star} > 0$, $\Lambda_s \to \infty$ and from (5.5), $\Lambda_b^{\star\star} < \infty$.

Equations (6.2) and (6.3) follow directly from taking the limit of (5.16) and (5.17) as $\lambda \to \infty$ when $L^{\star\star} \in (0, 1]$ and $\Lambda_s \to \infty$. Further, it follows from $(A.45)$ and $(A.51)$ that $\underline{x}^{\star\star} < 1, \overline{x}^{\star\star} < 1$ whenever $L^{\star\star} > 0$. If $L^{\star\star} = 1$, then from (6.2) and (6.3),

$$\frac{y_u}{y_u + y_d}a - 1 \geq \frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}} - 1\right) - 1\right) > 0.$$

If $L^{\star\star} \in (0,1)$, then from (6.2) and (6.3),

$$
\begin{aligned}
&\frac{y_u}{y_u+y_d}a-1\\
=&\frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L^{\star\star}}-1\right)-L^{\star\star}\right)-\frac{y_d}{y_u+y_d}(1-L^{\star\star})+\frac{(1-\alpha)y_d^2\xi}{(y_u+y_d)k}e^{-\frac{k}{r\xi}L^{\star\star}}\int_0^{\min\left\{\frac{(1-L^{\star\star})k}{(1-\alpha)y_d\xi}e^{\frac{k}{r\xi}L^{\star\star}},1\right\}}\frac{\rho(1-s)ds}{\rho-rs}\\
\leq&\frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}}-1\right)-1\right)-\frac{y_d}{y_u+y_d}\left(1-L^{\star\star}-\int_0^{\min\left\{1-L^{\star\star},(1-\alpha)\frac{y_d\xi}{k}e^{-\frac{k}{r\xi}L^{\star\star}}\right\}}\frac{(1-\alpha)\frac{y_d\xi}{k}e^{-\frac{k}{r\xi}L^{\star\star}}-z}{(1-\alpha)\frac{y_d\xi}{k}e^{-\frac{k}{r\xi}L^{\star\star}}-\frac{r}{\rho}z}dz\right)\\
\leq&\frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}}-1\right)-1\right)-\frac{y_d}{y_u+y_d}\left(1-L^{\star\star}-\int_0^{1-L^{\star\star}}\frac{(1-\alpha)\frac{y_d\xi}{k}e^{-\frac{k}{r\xi}L^{\star\star}}-z}{(1-\alpha)\frac{y_d\xi}{k}e^{-\frac{k}{r\xi}L^{\star\star}}-\frac{r}{\rho}z}dz\right)\\
\leq&\frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}}-1\right)-1\right)-\frac{y_d}{y_u+y_d}\left(1-L^{\star\star}-\int_0^{1-L^{\star\star}}\frac{(1-\alpha)\frac{y_d\xi}{k}e^{-\frac{k}{r\xi}L^{\star\star}}-z}{(1-\alpha)\frac{y_d\xi}{k}e^{-\frac{k}{r\xi}L^{\star\star}}-\frac{r}{\rho}z}dz\right)\\
\leq&\frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}}-1\right)-1\right).
\end{aligned}
$$

2b) Equation (6.4) follows directly from taking the limit of (5.17) as $\lambda \to \infty$ when $L^{\star\star} = 0$ and $\Lambda^{\star\star} < \infty$. It follows from $(A.45)$ and $(A.51)$ that $\underline{x}^{\star\star} = \overline{x}^{\star\star} = 1$. It follows from (5.5) and (5.16) that

$$\lim_{\lambda\to\infty}\Lambda_b = \lim_{\lambda\to\infty}\frac{k\rho}{\xi r}\frac{L}{e^{\frac{k}{\xi r}L}-1-\frac{k}{\xi r}L} = \frac{k\rho}{\xi r}\lim_{\lambda\to\infty}\frac{1}{\frac{k}{\xi r}(e^{\frac{k}{\xi r}L}-1)} = \infty,$$

where we used l'Hospital rule and $\lim_{\lambda\to\infty} L = 0$. Note that

$$y_d\int_0^{\min\left\{1,\frac{\rho+\Lambda_s^{\star\star}}{\Lambda_s^{\star\star}}\frac{k}{(1-\alpha)y_d\xi}\right\}}\frac{\Lambda_s^{\star\star}(1-s)}{y_u+y_d+\Lambda_s^{\star\star}\left(1-s+\frac{y_u+y_d}{\rho}s\right)}ds < y_d,$$

and so, if (6.4) has a solution, then $\frac{y_u}{y_u+y_d}a < 1$ and (6.4) does not have a solution when $\frac{y_u}{y_u+y_d}a \geq 1$. It follows from Lemma 3 that $|\Theta_I| = 1-|\Theta_L|-|\Theta_M| = 1-L-(\underline{\theta}-\check{\theta}) \to 1-\underline{\theta}^{\star\star}$ as $\lambda\to\infty$ and by $(A.48)$, $\underline{\theta}^{\star\star} = \min\{1,(1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s^{\star\star}}{\rho+\Lambda_s^{\star\star}}\} < \min\{1,\frac{y_d\xi}{k}\}$ which proves $|\Theta_I| > 1-\frac{y_d\xi}{k}$.

2c) The last statement follows from the fact that L is decreasing in λ proven in Proposition 7. $\square$

The following lemma shows that the expression for I in (5.18) is greatly simplified when $\Theta_I \neq \emptyset$, which allows for a clean derivation of the comparative statics that follows.

Lemma 18. *If $\Theta_I \neq \emptyset$, then*

$$I(L,\Lambda_s) = \frac{\Lambda_s L}{y_u+y_d+\Lambda_s} + (1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}e^{-\frac{k}{r\xi}L}\int_0^1\frac{1-s}{1+\frac{y_u+y_d}{\Lambda_s}-\frac{r}{\rho}s}ds. \tag{A.52}$$

Proof. If $\Theta_I \neq \emptyset$, then $\hat{\theta} > \underline{\theta}$ and so, by $(A.43)$ and $(A.48)$, $1 - L > (1-\alpha)\frac{y_d \xi}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{\frac{k}{r\xi}L}$ which implies $(A.52)$. $\square$

Proof of Proposition 3. Consider functions Λ_s^1 and Λ_s^2 introduced in the proof of Theorem 2. Consider an increase in y_d and a decrease in y_u so that $y_d + y_u$ does not change. This leads to an upward shift of Λ_s^1 and a downward shift of Λ_s^2. By Lemma 23, Λ_s^1 is increasing and Λ_s^2 is decreasing and so, the change in y_d and y_u leads to a decrease in L. $\square$

Bargaining Delays and Liquidity

Proof of Proposition 4. To get (7.2), plug functions c and v from $(A.27)$ and $(A.28)$ into (3.1) and then substitute $V_{bu}(\phi)$ and $V_{su}(\theta)$ from (5.14) and $(A.32)$. By the definition of the SBS, the first offers are accepted by the sellers of the lowest quality and buyers of the highest quality, thus, $ba = p(1) - p(0) = \frac{k}{r}$. $\square$

Proof of Proposition 5. Follows directly from Lemma 3 and (7.1). $\square$

Proof of Proposition 6. From $(A.43), (A.44)$, and $(A.48)$, $\hat{\theta}$,$\check{\theta}$ and $\underline{\theta}$ converge to $1 + \frac{r}{k}\xi \ln \underline{x}^{\dagger}$,0, and 0, resp. whenever $\hat{\theta}^{\dagger} > 0$ and all converge to zeros whenever $\hat{\theta}^{\dagger} = 0$. By definition, $x^{\dagger}(\theta) = 0$ for $\theta \in (0, \hat{\theta})$. By the equation $A.41$, $x(\cdot)$ on $[\hat{\theta}, 1]$ is decribed by $\frac{d}{d\theta}x^{\dagger}(\theta) = \frac{k}{\xi r \alpha}x^{\dagger}(\theta) > 0$ and so, $x(\cdot)$ is increasing on $(0, 1]$. $\square$

Proof of Proposition 7. By moving all terms to the right-hand side, we can rewrite the system (5.16) and (5.17) in

$$\begin{cases} 0 = & g_1(\Lambda_s, L, k), \\ 0 = & g_2(\Lambda_s, L, k), \end{cases} \tag{A.53}$$

where the first line corresponds to (5.16) and the second line to (5.17). From Lemma 23, $\frac{\partial g_1}{\partial \Lambda_s} < 0, \frac{\partial g_2}{\partial \Lambda_s} < 0, \frac{\partial g_1}{\partial L} > 0, \frac{\partial g_2}{\partial L} < 0$. Moreover, we can see that $\frac{\partial g_2}{\partial \Lambda_s} < \frac{1}{\lambda}$ and $\frac{\partial g_1}{\partial \Lambda_s} = -1$. Further, $\frac{\partial g_1}{\partial k} = \frac{\lambda \xi y_d r}{k^2 \rho}\left(1 + e^{\frac{k}{\xi r}L}\left(\frac{k}{\xi r}L - 1\right)\right) > 0$ and $\frac{\partial g_2}{\partial k} > (1-\alpha)C$ for some constant C. By the implicit function theorem,

$$\frac{dL}{dk} = \frac{-\frac{\partial g_1}{\partial \Lambda_s}\frac{\partial g_2}{\partial k} + \frac{\partial g_2}{\partial \Lambda_s}\frac{\partial g_1}{\partial k}}{\frac{\partial g_1}{\partial \Lambda_s}\frac{\partial g_2}{\partial L} - \frac{\partial g_2}{\partial \Lambda_s}\frac{\partial g_1}{\partial L}} \leq \frac{(1-\alpha)C - \frac{\xi y_d r^2}{k^2 \rho}\left(1 + e^{\frac{k}{\xi r}L}\left(\frac{k}{\xi r}L - 1\right)\right)}{\frac{\partial g_1}{\partial \Lambda_s}\frac{\partial g_2}{\partial L} - \frac{\partial g_2}{\partial \Lambda_s}\frac{\partial g_1}{\partial L}}.$$

The denominator of this upper bound on $\frac{dL}{dk}$ is positive and the numerator is negative for sufficiently large α. Thus, if $\alpha > 1 - \frac{\xi y_d r^2}{C k^2 \rho}\left(1 + e^{\frac{k}{\xi r}\tilde{L}}\left(\frac{k}{\xi r}\tilde{L} - 1\right)\right)$, then for all equilibria with $L > \tilde{L}$, $\frac{dL}{dk} < 0$. Further, $\frac{d\Lambda_s}{dk} = -\frac{\frac{\partial g_2}{\partial L}\frac{dL}{dk} + \frac{\partial g_2}{\partial k}}{\frac{\partial g_2}{\partial \Lambda_s}} > 0$.

To show that $\frac{d\underline{x}}{dk} < 0$, we use $(A.45)$ to express (5.16) and (5.17) in terms of $(\Lambda_s, \underline{x})$:

$$\begin{cases} 0 = \frac{\xi r \lambda y_d}{k\rho}\left(\frac{1}{\underline{x}} - 1 + \ln \underline{x}\right) - (y_u + y_d) - \Lambda_s, \\ 0 = \frac{y_u(a-1)}{y_u+y_d} - \frac{\Lambda_s}{\lambda} + \frac{y_d}{k(y_u+y_d)}\left(\frac{\Lambda_s}{y_u+y_d+\Lambda_s} r\xi \ln \underline{x} - (1-\alpha) y_d \xi \frac{\Lambda_s \underline{x}}{\rho+\Lambda_s}\int_0^1 \frac{1-s}{1+\frac{y_u+y_d}{\Lambda_s} - \frac{r}{\rho}s} ds\right), \end{cases} \tag{A.54}$$

We write f_1 for the first equation and f_2 for the second. By the implicit function theorem, $\frac{d\underline{x}}{dk} = \frac{-\frac{\partial f_1}{\partial \Lambda_s}\frac{\partial f_2}{\partial k} + \frac{\partial f_2}{\partial \Lambda_s}\frac{\partial f_1}{\partial k}}{\frac{\partial f_1}{\partial \Lambda_s}\frac{\partial f_2}{\partial \underline{x}} - \frac{\partial f_2}{\partial \Lambda_s}\frac{\partial f_1}{\partial \underline{x}}}$. From Lemma 23, $\frac{\partial f_1}{\partial \Lambda_s} < 0, \frac{\partial f_2}{\partial \Lambda_s} < 0, \frac{\partial f_1}{\partial \underline{x}} < 0, \frac{\partial f_2}{\partial \underline{x}} > 0$. Moreover, $\frac{\partial f_2}{\partial k} > 0, \frac{\partial f_1}{\partial k} < 0$ and so, $\frac{d\underline{x}}{dk} < 0$. From $(A.45)$ and $(A.51)$, $\overline{x} = \frac{1-\underline{x}}{-\ln \underline{x}}$ is increasing in $\underline{x}$, and so, $\frac{d\overline{x}}{dk} < 0$. Since $X = \overline{x}L$, $\frac{dX}{dk} < 0$.

To derive the comparative statics in λ, we express equilibrium conditions (5.16) and (5.17) in terms of variables L and $M_{bu}(\phi)$ (which we shortly denote by M_ϕ below) as follows[40]

$$\begin{cases} M_\phi = \frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L} - 1\right) - L\right) - \frac{y_u+y_d}{\lambda}, \\ M_\phi = \frac{y_u(a-1)}{y_u+y_d} - \frac{y_d}{y_u+y_d}\left(\frac{\lambda M_\phi L}{y_u+y_d+\lambda M_\phi} + (1-\alpha)\frac{y_d \xi}{k}\frac{\lambda M_\phi}{\rho+\lambda M_\phi} e^{-\frac{k}{r\xi}L}\int_0^1 \frac{1-s}{1+\frac{y_u+y_d}{\lambda M_\phi} - \frac{r}{\rho}s} ds\right). \end{cases} \tag{A.55}$$

Again, we write h_1 for the first equation and h_2 for the second. By the implicit function theorem, $\frac{dL}{d\lambda} = \frac{-\frac{\partial h_1}{\partial M_\phi}\frac{\partial h_2}{\partial \lambda} + \frac{\partial h_2}{\partial M_\phi}\frac{\partial h_1}{\partial \lambda}}{\frac{\partial h_1}{\partial M_\phi}\frac{\partial h_2}{\partial L} - \frac{\partial h_2}{\partial M_\phi}\frac{\partial h_1}{\partial L}}$. From Lemma 23, $\frac{\partial h_1}{\partial M_\phi} = -1, \frac{\partial h_2}{\partial M_\phi} < -1, \frac{\partial h_1}{\partial L} > 0, \frac{\partial h_2}{\partial L} < 0$ and so, the denominator is positive. Moreover, $\frac{\partial h_2}{\partial \lambda} < 0$ and $\frac{\partial h_1}{\partial \lambda} = \frac{y_u+y_d}{\lambda^2} > 0$ and so, $-\frac{\partial h_1}{\partial M_\phi}\frac{\partial h_2}{\partial \lambda} + \frac{\partial h_2}{\partial M_\phi}\frac{\partial h_1}{\partial \lambda} \leq \frac{\partial h_2}{\partial \lambda} - \frac{y_u+y_d}{\lambda^2} < 0$. Thus, $\frac{dL}{d\lambda} < 0$. By $(A.45)$ and $\overline{x} = \frac{1-\underline{x}}{-\ln \underline{x}}$, $\frac{d\underline{x}}{d\lambda} > 0$ and $\frac{d\overline{x}}{d\lambda} > 0$. Further, by applying the implicit function theorem to $(A.53)$,

$$\frac{d\Lambda_s}{d\lambda} = \frac{-\frac{\partial g_1}{\partial L}\frac{\partial g_2}{\partial \lambda} + \frac{\partial g_2}{\partial L}\frac{\partial g_1}{\partial \lambda}}{\frac{\partial g_1}{\partial L}\frac{\partial g_2}{\partial \Lambda_s} - \frac{\partial g_2}{\partial L}\frac{\partial g_1}{\partial \Lambda_s}}.$$

From Lemma 23, $\frac{\partial g_1}{\partial \Lambda_s} < 0, \frac{\partial g_2}{\partial \Lambda_s} < 0, \frac{\partial g_1}{\partial L} > 0, \frac{\partial g_2}{\partial L} < 0$ and so, the denominator is negative. Moreover, $\frac{\partial g_1}{\partial \lambda} > 0, \frac{\partial g_2}{\partial \lambda} = \frac{\Lambda_s}{\lambda^2} > 0$ and so, $\frac{d\Lambda_s}{d\lambda} > 0$. □

Model with Two Asset Classes

Lemma 19. *An increase in a leads to an increase in L,X and Λ_s and a decrease in $\underline{x}$.*

Proof. An increase in a leads to an upward shift of Λ_s^2 and so, an increase in Λ_s and L. By $(A.45)$ and $(A.51)$, it leads to a decrease in $\underline{x}$ and increase in X. □

Lemma 20. *There exists a unique two-class equilibrium.*

[40]Again we consider only cases where before and after an increase in λ, $L < 1$, as other cases are straightforward to show from (5.4).

Proof. Equilibrium quantities $(\Lambda_{s,1}, \underline{x}_1)$ and $(\Lambda_{s,2}, \underline{x}_2)$ in two classes are determined by the unique solution to the system $(A.54)$ with $a = a_1$ and $a = a_2$, respectively. Denote by $\underline{x}(a)$ the equilibrium threshold of the buyer's strategy given that the mass of investors is a. Equations in the system $(A.54)$ are continuous in parameters and so, the solution $(\Lambda_s, \underline{x})$ varies continuously with a and an increase in a leads to a decrease in $\underline{x}$ by Lemma 19. Thus, $\underline{x}(\cdot)$ is continuous and decreasing in a. By (8.1), a_1 is determined by $\underline{x}(a_1) = \underline{x}(a - a_1)$ which has a unique solution. $\square$

Proof of Proposition 8. Suppose k_1 increases. We show that as a result a_1 decreases and a_2 increases. Suppose to contradiction that a_1 increases and a_2 decreases. By Propositions 7 and Lemma 19, $\underline{x}_1$ decreases and $\underline{x}_2$ increases which contradicts the indifference of buyers (8.1). By Lemma 19, the drop in L_1 resulting from a decrease in k_1 is larger because of the additional effect of the increase in a_1. The argument for an increase in λ_2 and a decrease in k_2 is analogous. $\square$

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Online Appendix (Not for Publication)

A.3 Flow Payoff Specification for Fixed-Income Assets

Consider the infinite-maturity bond with the coupon 1 paid at a constant rate c_0. The issuer defaults on the bond with at a constant rate $c_1(1-\theta)$. In the case of default, the bond-holder incurs costs c_2, and the bond is immediately reissued to the same holder after the default. Thus, the flow payoff from holding this security is $c_0 - c_1 c_2(1-\theta) = c_0 - c_1 c_2 + c_1 c_2 \theta$ which gives the flow payoff specification in this paper.[41]

A.4 Example of F

Here, we provide an example of a sequence of F approximating F^* that satisfies our assumptions. Fix $\gamma > 0$. Suppose θ normally distributed with zero mean and variance $\gamma^2 - \frac{1}{\gamma}$, and ε_b and ε_s are independent normals with zero mean and variance $\frac{1}{\gamma}$. Let F be the distribution of $(\theta + \varepsilon^s, \theta + \varepsilon^b)$ conditional on $(\theta + \varepsilon^s, \theta + \varepsilon^b) \in [0,1]$.

Proposition 9. *1. F is affiliated;*

2. $F \xrightarrow{p} F^*$ as $\gamma \to \infty$;

3. for any $\varepsilon > 0$, there is $\overline{\gamma}$ so that for all $\gamma > \overline{\gamma}$,

$$\sup_{(\theta^s,\theta^b):|\theta^s-\theta^b|>\varepsilon} \max\{f(\theta^b|\theta^s), f(\theta^s|\theta^b)\} < \varepsilon. \tag{A.56}$$

Proof. 1) By definition, F is a bivariate normal distribution with zero mean and covariance matrix $\Sigma = \begin{pmatrix} \gamma^2 & \gamma^2 - \frac{1}{\gamma} \\ \gamma^2 - \frac{1}{\gamma} & \gamma^2 \end{pmatrix}$ conditional on $(\theta^s, \theta^b) \in [0,1]^2$. Since the density of the positively correlated bivariate normal distribution is log-supermodular so is f. Thus, F is affiliated.

2) The densify f is given by

$$\begin{aligned} f(\theta^s, \theta^b) &= \frac{\exp\left(-\frac{(\theta^s)^2 + (\theta^b)^2 - 2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)}{\int_0^1 \int_0^1 \exp\left(-\frac{(\theta^s)^2 + (\theta^b)^2 - 2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right) d\theta^s d\theta^b} \\ &= \frac{\exp\left(-\frac{\gamma(\theta^s-\theta^b)^2 + 2\gamma^{-2}\theta^s\theta^b}{2(2-\gamma^{-3})}\right)}{\int_0^1 \int_0^1 \exp\left(-\frac{\gamma(\theta^s-\theta^b)^2 + 2\gamma^{-2}\theta^s\theta^b}{2(2-\gamma^{-3})}\right) d\theta^s d\theta^b}. \end{aligned}$$

[41] Recall that the shift by the constant of flow payoffs will change only asset prices, but not affect the results about the asset liquidity (see footnote 13).

After the change of variables $x = \theta^s - \theta^b, y = \theta^s\theta^b$, we have

$$f(x,y) = \frac{\exp\left(-\frac{\gamma x^2 + 2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)}{2\int_0^1\int_0^{1-x}\exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)\frac{dydx}{\sqrt{x^2+4y}}}.$$

We next construct upper and lower bounds on the nominator and the denominator of f. For the nominator, the bounds are

$$\exp\left(-\tfrac{1}{3}\gamma x^2 - \tfrac{2}{3}\gamma^{-2}\right) \le \exp\left(-\frac{\gamma x^2 + 2\gamma^{-2}y}{2(2-\gamma^{-3})}\right) \le \exp\left(-\tfrac{1}{5}\gamma x^2\right).$$

For the denominator, for any $\varepsilon_0 \in (0, \frac{1}{2})$ the upper bound is

$$\begin{aligned}
\int_0^1\int_0^{1-x}\exp\left(-\frac{\gamma x^2 + 2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)\frac{dydx}{\sqrt{x^2+4y}} &\le \int_0^1\int_0^{1-x}\exp\left(-\frac{\gamma x^2}{5}\right)\frac{dydx}{\sqrt{x^2+4y}}\\
&= \int_0^1 \exp\left(-\tfrac{1}{5}\gamma x^2\right)(1-x)dx\\
&\le \int_0^1 \exp\left(-\tfrac{1}{5}\gamma x^2\right)dx\\
&\le \exp\left(-\tfrac{1}{5}\gamma^{2\varepsilon_0}\right)\left(1-\gamma^{-1/2+\varepsilon_0}\right) + \gamma^{-1/2+\varepsilon_0}\\
&\le c_1\gamma^{-1/2+\varepsilon_0},
\end{aligned}$$

and the lower bound is

$$\begin{aligned}
\int_0^1\int_0^{1-x}\exp\left(-\frac{\gamma x^2 + 2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)\frac{dydx}{\sqrt{x^2+4y}} &\ge \int_0^1\int_0^{1-x}\exp\left(-\tfrac{1}{3}\gamma x^2 - \tfrac{2}{3}\gamma^{-2}\right)\frac{dydx}{\sqrt{x^2+4y}}\\
&= \int_0^1 \exp\left(-\tfrac{1}{3}\gamma x^2 - \tfrac{2}{3}\gamma^{-2}\right)(1-x)dx\\
&\ge \gamma^{-1/2}\exp\left(-\tfrac{2}{3}\right)\left(1-\gamma^{-1/2}\right)\\
&\ge c_2\gamma^{-1/2}.
\end{aligned}$$

Thus,

$$\frac{1}{c_1}\gamma^{1/2-\varepsilon}\exp\left(-\tfrac{1}{3}\gamma x^2 - \tfrac{2}{3}\gamma^{-2}\right) \le \quad f(x,y) \quad \le \frac{1}{c_2}\gamma^{1/2}\exp\left(-\tfrac{1}{5}\gamma x^2\right),$$

and so,

- for all $|x| > \gamma^{-1/4}$, $f(x,y) < \frac{1}{c_2}\gamma^{1/2}\exp\left(-\frac{1}{5}\gamma^{1/2}\right) \underset{\gamma\to\infty}{\to} 0$;
- for all $|x| < \gamma^{-1}$, $f(x,y) > \frac{1}{c_1}\gamma^{1/2-\varepsilon_0}\exp\left(-\frac{1}{3}\gamma^{-1} - \frac{2}{3}\gamma^{-2}\right) \underset{\gamma\to\infty}{\to} \infty$;

- for any x,

$$1 \leq \frac{\max_{y\in[0,|x|]} f(x,y)}{\min_{y\in[0,|x|]} f(x,y)} = \frac{\max_{y\in[0,|x|]} \exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)}{\min_{y\in[0,|x|]} \exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)} \leq \exp\left(\frac{\gamma^{-2}}{2-\gamma^{-3}}\right) \underset{\gamma\to\infty}{\to} 1.$$

This implies that $F \xrightarrow{p} F^*$ as $\gamma \to \infty$.

3) For $|\theta^s - \theta^b| > \gamma^{-1/4}$,

$$\begin{aligned}
f(\theta^s|\theta^b) = \frac{f(\theta^s,\theta^b)}{\int_0^1 f(\theta^s,\theta^b)d\theta^s} &= \frac{\exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)}{\int_0^1 \exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)d\theta^s} \\
&\leq \frac{\exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)}{\int_{\max\{0,\theta^b-\gamma^{-1/4}\}}^{\min\{1,\theta^b+\gamma^{-1/4}\}} \exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)d\theta^s} \\
&\leq \frac{\frac{1}{c_2}\gamma^{1/2}\exp\left(-\frac{1}{5}\gamma(\theta^s-\theta^b)^2\right)}{\frac{2}{c_1}\gamma^{1/2-\varepsilon_0}\exp\left(-\frac{1}{3}\gamma^{-1/2}-\frac{2}{3}\gamma^{-2}\right)\gamma^{-1/4}} \\
&= \frac{c_1}{2c_2}\gamma^{1/4+\varepsilon_0}\exp\left(-\frac{1}{5}\gamma x^2+\frac{1}{3}\gamma^{-1/2}+\frac{2}{3}\gamma^{-2}\right) \\
&\leq \frac{c_1}{2c_2}\gamma^{1/4+\varepsilon_0}\exp\left(-\frac{1}{5}\gamma^{1/2}+\frac{1}{3}\gamma^{-1/2}+\frac{2}{3}\gamma^{-2}\right) \underset{\gamma\to\infty}{\to} 0,
\end{aligned}$$

and the symmertic argument holds for $f(\theta^b|\theta^s)$. Thus, $(A.56)$ obtains. □

A.5 Auxiliary Steps in the Proof of Theorem 2

Lemma 21. *Equation* (5.4) *implies equation* (5.17).

Proof. Using $(A.49)$ to substitute $\sigma(\theta)$ in equation (5.4), we get

$$\frac{\Lambda_s}{\lambda} = \frac{y_u}{y_u+y_d}(a-1) - \frac{y_d}{y_u+y_d}\left(\frac{\Lambda_s L}{y_u+y_d+\Lambda_s} + \int_{\check{\theta}}^{\theta} \frac{\Lambda_s\sigma(\theta)}{y_u+y_d+\Lambda_s\sigma(\theta)}d\theta\right).$$

To prove (5.17), we show that

$$\int_{\check{\theta}}^{\overline{\theta}} \frac{\Lambda_s\sigma(\theta)}{y_u+y_d+\Lambda_s\sigma(\theta)}d\theta = (1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}e^{-\frac{k}{r\xi}L}\int_0^{\min\left\{\frac{\rho+\Lambda_s}{\Lambda_s}\frac{(1-L)k}{(1-\alpha)y_d\xi}e^{\frac{k}{r\xi}L},1\right\}} \frac{1-s}{1+\frac{y_u+y_d}{\Lambda_s}-\frac{r}{\rho}s}ds. \tag{A.57}$$

Expressing $\sigma(\theta)$ from $(A.49)$, $\check{\theta}$ from $(A.44)$, and $\overline{\theta}$ from $(A.48)$, we get

$$\begin{aligned}
&\int_{\check{\theta}}^{\overline{\theta}} \frac{\Lambda_s\sigma(\theta)}{y_u+y_d+\Lambda_s\sigma(\theta)}d\theta \\
=&\int_{\check{\theta}}^{\check{\theta}+\min\left\{1+\frac{r\xi}{k}\ln\underline{x},(1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}\right\}} \frac{(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}-\frac{k}{y_d}(\theta-\check{\theta})}{\frac{y_u+y_d}{\rho}((1-\alpha)\frac{\rho}{\rho+\Lambda_s}\xi\underline{x}+\frac{k}{y_d}(\theta-\check{\theta}))+(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}-\frac{k}{y_d}(\theta-\check{\theta})}d\theta \\
=&\int_{\check{\theta}}^{\check{\theta}+\min\left\{1+\frac{r\xi}{k}\ln\underline{x},(1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}\right\}} \frac{(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}-\frac{k}{y_d}(\theta-\check{\theta})}{\left(1+\frac{y_u+y_d}{\Lambda_s}\right)(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}-\frac{r}{\rho}\frac{k}{y_d}(\theta-\check{\theta})}d\theta \\
=&y_d(1-\alpha)\frac{\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\underline{x}\int_0^{\min\left\{\frac{\rho+\Lambda_s}{\Lambda_s}\frac{\left(1+\frac{r\xi}{k}\ln\underline{x}\right)k}{(1-\alpha)y_d\xi\underline{x}},1\right\}}\frac{1-s}{1+\frac{y_u+y_d}{\Lambda_s}-\frac{r}{\rho}s}ds
\end{aligned}$$

where in the second line we make a change of variables $s=\frac{\theta-\check{\theta}}{(1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\underline{x}}$. After expressing $\underline{x}$ from $(A.45)$ we get equation $(A.57)$. □

Lemma 22. $\Lambda_s^2(\cdot)$ *is strictly decreasing.*

Proof. $\Lambda_s^2(L)$ is given implicitly by the equation (5.17). It is convenient to rewrite function I as follows

$$I(L,\Lambda_s)=\frac{\Lambda_s L}{y_u+y_d+\Lambda_s}+\int_0^{\underline{\theta}-\check{\theta}}\frac{(1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}e^{-\frac{k}{r\xi}L}-s}{\frac{\Lambda_s+y_u+y_d}{\Lambda_s}(1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}e^{-\frac{k}{r\xi}L}-\frac{r}{\rho}s}ds. \tag{A.58}$$

By the implicit function theorem,

$$\frac{d\Lambda_s^2}{dL}=\frac{-\frac{y_d}{y_u+y_d}\frac{\partial}{\partial L}I(L,\Lambda_s)}{\frac{1}{\lambda}+\frac{y_d}{y_u+y_d}\frac{\partial}{\partial\Lambda_s}I(L,\Lambda_s)}$$

Thus, to prove lemma it suffices to show that $\frac{\partial}{\partial\Lambda_s}I(L,\Lambda_s)>0$ and $\frac{\partial}{\partial L}I(L,\Lambda_s)>0$.

We first show $\frac{\partial}{\partial\Lambda_s}I(L,\Lambda_s)>0$. The first term in $(A.58)$ is strictly increasing in Λ_s. Let us consider the integrand in $(A.58)$. Denote $c = (1-\alpha)\frac{y_d \xi}{k} e^{-\frac{k}{r\xi}L}$. Then

$$\begin{aligned}
&\text{sgn}\frac{\partial}{\partial \Lambda_s}\left(\frac{\Lambda_s c - s(\rho+\Lambda_s)}{(\Lambda_s + y_u + y_d)c - \frac{r}{\rho}s(\rho+\Lambda_s)}\right) \\
=&\text{sgn}\frac{\partial}{\partial \Lambda_s}\left(\frac{\Lambda_s(c-s) - s\rho}{\Lambda_s(c-\frac{r}{\rho}s) + (y_u+y_d)c - rs}\right) \\
=&\text{sgn}\left((c-s)(\Lambda_s(c-\frac{r}{\rho}s) + (y_u+y_d)c - rs) - (c-\frac{r}{\rho}s)(\Lambda_s(c-s) - s\rho)\right) \\
=&\text{sgn}\left((c-s)((y_u+y_d)c - rs) + (c-\frac{r}{\rho}s)s\rho\right) \\
=&\text{sgn}\left((y_u+y_d)c^2 - (y_u+y_d)cs - rcs + cs\rho\right) \\
=&\text{sgn}\left(c^2(y_u+y_d)\right) = 1,
\end{aligned}$$

and so, the integrand is strictly increasing in Λ_s. Finally, $\underline{\theta} - \check{\theta} = \min\{1-L, (1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}e^{-\frac{k}{r\xi}L}\}$ is weakly increasing in Λ_s. Therefore, $\frac{\partial}{\partial \Lambda_s} I(\Lambda_s, L) > 0$.

We next show $\frac{\partial}{\partial L} I(L, \Lambda_s) > 0$. Let us introduce functions

$$\begin{aligned}
A(L, \Lambda_s) &= (1-\alpha)\frac{y_d\xi}{k}\frac{\Lambda_s}{\rho+\Lambda_s}e^{-\frac{k}{r\xi}L}, \\
b(\Lambda_s) &= \frac{\Lambda_s + y_u + y_d}{\Lambda_s}.
\end{aligned}$$

To simplify expression, we will omit arguments of A and b below. We can rewrite function I as follows

$$I(L, \Lambda_s) = \frac{L}{b} + \int_0^{\underline{\theta}-\check{\theta}} \frac{A - s}{bA - \frac{r}{\rho}s} ds. \tag{A.59}$$

Claim 7. *$\frac{1}{b} + \frac{\partial A}{\partial L}\frac{\rho}{r} > 0$ implies $\frac{\partial}{\partial L} I(L, \Lambda_s) > 0$.*

Proof of Claim 7: Differentiating $(A.59)$,

$$\frac{\partial}{\partial L} I(L, \Lambda_s) = \frac{1}{b} + \frac{\partial(\underline{\theta}-\check{\theta})}{\partial L}\left(\frac{A - (\underline{\theta}-\check{\theta})}{bA - \frac{r}{\rho}(\underline{\theta}-\check{\theta})}\right) + \frac{\partial A}{\partial L}\int_0^{\underline{\theta}-\check{\theta}} \frac{\partial}{\partial A}\left(\frac{A - \frac{r}{\rho}s}{bA - \frac{r}{\rho}s}\right) ds. \tag{A.60}$$

There are two cases. First, suppose $\underline{\theta} = \hat{\theta}$ and so, $\underline{\theta} - \check{\theta} = 1 - L$. Then

$$\begin{aligned}
\frac{\partial}{\partial L} I(L, \Lambda_s) =& \frac{1}{b} - \frac{A - (1-L)}{bA - \frac{r}{\rho}(1-L)} + \frac{\partial A}{\partial L}\int_0^{1-L} \frac{\partial}{\partial A}\left(\frac{A - \frac{r}{\rho}s}{bA - \frac{r}{\rho}s}\right) ds \\
=& \frac{(b - \frac{r}{\rho})(1-L)}{b(bA - \frac{r}{\rho}(1-L))} + \frac{\partial A}{\partial L}\int_0^{1-L} \frac{(b-\frac{r}{\rho})s}{(bA - \frac{r}{\rho}s)^2} ds.
\end{aligned}$$

Using the change of variables $z = bA - \frac{r}{\rho}s$, we can compute the integral as follows

$$\begin{aligned}
\int_0^{1-L} \frac{(b-\frac{r}{\rho})s}{(bA-\frac{r}{\rho}s)^2}ds &= (b-\frac{r}{\rho})\frac{\rho}{r}\int_0^{1-L}\left(\frac{bA}{(bA-\frac{r}{\rho}s)^2} - \frac{1}{(bA-\frac{r}{\rho}s)}\right)ds \\
&= (b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2\int_{bA}^{bA-\frac{r}{\rho}(1-L)}\left(-\frac{bA}{z^2}+\frac{1}{z}\right)dz \\
&= (b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2\int_{bA}^{bA-\frac{r}{\rho}(1-L)} d\left(\frac{bA}{z}+\ln z\right) \\
&= (b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2\left(\frac{bA}{bA-\frac{r}{\rho}(1-L)} - 1 + \ln\left(\frac{bA-\frac{r}{\rho}(1-L)}{bA}\right)\right) \\
&= (b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2\left(\frac{\frac{r}{\rho}(1-L)}{bA-\frac{r}{\rho}(1-L)} + \ln\left(\frac{bA-\frac{r}{\rho}(1-L)}{bA}\right)\right)
\end{aligned}$$

Since $\frac{\partial A}{\partial L} = -\frac{k}{r\xi}A < 0$,

$$\begin{aligned}
\frac{\partial}{\partial L}I(L,\Lambda_s) =&\frac{(b-\frac{r}{\rho})(1-L)}{b(bA-\frac{r}{\rho}(1-L))} + \frac{\partial A}{\partial L}(b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2\left(\frac{\frac{r}{\rho}(1-L)}{bA-\frac{r}{\rho}(1-L)} + \ln\left(\frac{bA-\frac{r}{\rho}(1-L)}{bA}\right)\right) \geq \\
\geq&\frac{(b-\frac{r}{\rho})(1-L)}{b(bA-\frac{r}{\rho}(1-L))} + \frac{\partial A}{\partial L}(b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2\frac{\frac{r}{\rho}(1-L)}{bA-\frac{r}{\rho}(1-L)} = \\
=&\frac{(b-\frac{r}{\rho})(1-L)}{bA-\frac{r}{\rho}(1-L)}\left(\frac{1}{b}+\frac{\partial A}{\partial L}\frac{\rho}{r}\right).
\end{aligned}$$

Now, suppose $\underline{\theta} < \hat{\theta}$ and so $\underline{\theta} = \check{\theta} + A$. Then $(A.60)$ becomes

$$\begin{aligned}
\frac{\partial}{\partial L}I(L,\Lambda_s) &= \frac{1}{b} + \frac{\partial A}{\partial L}\int_0^A \frac{\partial}{\partial A}\left(\frac{A-s}{bA-\frac{r}{\rho}s}\right)ds \\
&= \frac{1}{b} + \frac{\partial A}{\partial L}\int_0^A \frac{(b-\frac{r}{\rho})s}{(bA-\frac{r}{\rho}s)^2}ds.
\end{aligned}$$

Again, using the change of variables $z = bA - \frac{r}{\rho}s$, we can compute the integral as follows

$$\begin{aligned}
\int_0^A \frac{(b-\frac{r}{\rho})s}{(bA-\frac{r}{\rho}s)^2}ds &= (b-\frac{r}{\rho})\frac{\rho}{r}\int_0^A \left(\frac{bA}{(bA-\frac{r}{\rho}s)^2} - \frac{1}{(bA-\frac{r}{\rho}s)}\right)ds \\
&= (b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2 \int_{bA}^{(b-\frac{r}{\rho})A}\left(-\frac{bA}{z^2}+\frac{1}{z}\right)dz \\
&= (b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2 \int_{bA}^{(b-\frac{r}{\rho})A} d\left(\frac{bA}{z}+\ln z\right) \\
&= (b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2 \left(\frac{b}{b-\frac{r}{\rho}} - 1 + \ln\left(\frac{b-\frac{r}{\rho}}{b}\right)\right) \\
&= (b-\frac{r}{\rho})\left(\frac{\rho}{r}\right)^2 \left(\frac{\frac{r}{\rho}}{b-\frac{r}{\rho}} + \ln\left(\frac{b-\frac{r}{\rho}}{b}\right)\right)
\end{aligned}$$

Since $\frac{\partial A}{\partial L} = -\frac{k}{r\xi}A < 0$,

$$\frac{\partial}{\partial L}I(L,\Lambda_s) = \frac{1}{b} + \frac{\partial A}{\partial L}(b-\frac{r}{\rho})(\frac{\rho}{r})^2\left(\frac{\frac{r}{\rho}}{b-\frac{r}{\rho}} + \ln\left(\frac{b-\frac{r}{\rho}}{b}\right)\right) \geq \frac{1}{b} + \frac{\partial A}{\partial L}\frac{\rho}{r}.$$

□

Claim 8. $1-\alpha < \frac{r^2}{y_d(r+y_u+y_d)}$ *implies* $\frac{1}{b} + \frac{\partial A}{\partial L}\frac{\rho}{r} > 0$.

Proof of Claim 8:

$$\begin{aligned}
\frac{1}{b} + \frac{\partial A}{\partial L}\frac{\rho}{r} &= \frac{\Lambda_s}{y_u+y_d+\Lambda_s} - (1-\alpha)\frac{y_d\rho}{r^2}\frac{\Lambda_s}{\rho+\Lambda_s}e^{-\frac{k}{r\xi}L} \\
&\geq \frac{\Lambda_s}{\rho+\Lambda_s}\left(1-(1-\alpha)\frac{y_d\rho}{r^2}e^{-\frac{k}{r\xi}L}\right) \\
&\geq \frac{\Lambda_s}{\rho+\Lambda_s}\frac{r^2-(1-\alpha)y_d\rho}{r^2}
\end{aligned}$$

which is positive whenever $1-\alpha < \frac{r^2}{y_d(r+y_u+y_d)}$. *q.e.d.*

To summarize, $1-\alpha < \frac{r^2}{y_d(r+y_u+y_d)}$ implies that $\frac{\partial}{\partial L}I(L,\Lambda_s) > 0$.

Lemma 23. *If* $1-\alpha < \frac{r^2}{y_d(r+y_u+y_d)}$, *then there is a unique solution* (Λ_s, L) *to* (5.16) *and* (5.17).

Proof. By Lemma 22, $\Lambda_s^2(\cdot)$ solving (5.17) is strictly decreasing. The right-hand of equation (5.16) is strictly increasing in L, as $\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L}-1\right)-L\right)'_L = e^{\frac{k}{\xi r}L}-1>0$. Thus, Λ_s^1 is strictly increasing. Therefore, the solution to (5.16) and (5.17) is unique. □